Inverse Problems for Parabolic Integro-Differential Equations with Instant and Integral Conditions

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 Declaration: Hereby I declare that this doctoral thesis, my original investigation and achievement, submitted for the doctoral degree at Tallinn University of Technology has not been submitted for any doctoral or equivalent academic degree.

 Kairi Kasemets

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Applicant’s contribution in these publications

I  Performing technical computations of the proofs, participation in writing the manuscript.

II  Performing the proofs, participation in writing the manuscript.

III  Contributing in ideas of proofs and formulation of statements, performing the proofs, preparing the manuscript.

In addition to I - III, the following publication related to the topic of the thesis has appeared:


Publication II contains results that generalize results of IIa. Therefore, Publication IIa doesn’t form a basis of the present thesis.
INTRODUCTION

There are several definitions of direct and inverse problems. In physics, the direct problem means the determination of states using model parameters and the inverse problem means the determination of parameters of models using given states. Often the model is described by differential or integro-differential equations. Then the direct problem means the solution of the equation subject to proper boundary and/or initial conditions. In this narrower mathematical sense, the inverse problem means the identification of parameters of the equation (e.g. coefficients, free terms, kernels) or boundary or initial conditions on the basis of information available on the solution of the direct problem. Different mathematical and physical aspects of inverse problems for partial differential equations can be found in monographs and articles [2, 8, 20, 22, 23, 26, 38, 39, 51, 59].

An important issue is the well-posedness of a posed problem. A problem is well-posed in the sense of Hadamard [17] if the solution exists, is unique and continuously depends on the data (the latter one is the so-called stability requirement). In case the problem is not well-posed, it is called ill-posed. In case of nonexistence the problem is over-determined and contradictory. The non-uniqueness or non-stability indicates lack of information. Provided the solution exist, usually it is possible to find functional spaces where it is unique and stable. However, for an inverse problem those spaces often contain derivatives of data, which are not directly measurable. This means ill-posedness from the practical viewpoint. In case the spaces where the problem is well-posed contain derivatives up to some finite order from the data then the problem is called moderately ill-posed. The highest degree of the derivative involved in such a space is called the degree of ill-posedness of the problem. In case such spaces involve all derivatives of the data, the problem is called severely ill-posed. To solve ill-posed problems, regularization techniques are used [9, 19, 38, 62].

Starting from ca 1970 models with memory to describe heat processes were introduced and developed [6, 16, 36, 37, 49, 50, 52] (see also the monographs [1, 57]). In those models the temperature satisfies parabolic integro-differential equations that contain integral terms with kernels related to the "memory" of the material. Incorporation of memory terms brings along an inertia to the heat process and such models are more relevant from the practical viewpoint. In parallel, models with memory were introduced for mechanical processes, too (viscoelastic materials), leading to hyperbolic integro-differential equations [10, 58].

The study of inverse problems for parabolic and hyperbolic integro-differential equations with memory terms started in the middle of 1980s. First series of papers [12, 14, 24, 30, 31, 32, 33, 34, 48, 64, 66] was devoted to the identification of time-dependent kernels using information about certain
traces of the solutions of the direct problems over the time. Such problems are moderately ill-posed and can be reduced to Volterra equations of the second kind. In case the parabolic equation is linear, the corresponding Volterra equation contains nonlinearities that are only of convolution type. In [24] a method of norms with exponential weights was proposed to prove global (in time) existence and stability of the solutions of such problems. This method was exploited in many subsequent papers.

Approximately in the middle of 1990s the study of inverse problems to determine space- and time-dependent memory kernels in parabolic integro-differential equations started. One class of treated problems consists in identification of kernels that depend only on some part of the space variables or have radial or else symmetries under information about traces of the solutions of the direct problems over the time, again [7, 11, 28]. Into this group of results we can put also papers dealing with determination of kernels representable in the form of finite sums of products of known space-dependent and unknown time-dependent functions [35, 53, 54, 55]. It turns out that those problems still admit the reduction to Volterra equations of the second kind and can be treated as before, in particular the method of weighted norms enables to prove global existence and stability.

In case the kernel depends on all space variables, the inverse problem may be posed on the basis of the Dirichlet-to-Neumann map. The uniqueness of the solution of such a problem was proved in [25]. The proof adjusts the celebrated method of Sylvester and Uhlmann [61] to the integro-differential case.

Another direction is the treatment of problems to determine other space-dependent parameters than the memory kernels. In case the unknown parameter of the equation depends only on space variables, it is natural to use an additional information of the same structure in the inverse problem, e.g. traces of solutions of direct problems at fixed time values (instant conditions) or integrals over fixed time domains (integral conditions). Those problem are not of Volterra type any more. For example, the problem to determine a space-dependent free term in a usual parabolic equation by means of final over-determination of the solution can be reduced to a Fredholm equation of the second kind [21]. There are two possibilities to handle such type of problems. One way consists in applying the fixed point argument under certain smallness restrictions (local results). This was exploited in papers [45, 47]. Another way is to avoid smallness restrictions and to apply the Fredholm alternative. Then the uniqueness implies the existence and stability. Actually, the latter one is the starting point of the investigations of the present thesis.

The first aim is to prove uniqueness of the solution of the inverse problem to determine a space-dependent component of a source term of a parabolic integro-differential equation in case the solution of the direct
problem is over-determined at the final moment of time (problem IP1) without assuming smallness restrictions (global result). This was previously proved for the usual parabolic equation in [21] (see also [4] for the semi-linear case). The proof uses a positivity principle for the direct problem (e.g. positivity of data implies the positivity of the solution) that is an immediate consequence of the well-know maximum principle. Thus, in the first step, we prove such a principle for parabolic integro-differential equations. Further, by means of the positivity principle the uniqueness of the mentioned inverse problem is shown. The assumptions contain certain positivity and monotonicity restrictions on the time-dependent component of the source term. Making use of the proved uniqueness and Fredholm-type results for an analogous problem for the usual parabolic results, we prove the existence and stability of the solution of the inverse problem.

We mention that the existence and stability was previously proved in [46] under the assumption that the memory kernel is positive and the solution of inverse problem is unique. We do not need the positivity of the kernel in the Fredholm-type result.

The next aim is to prove the global uniqueness and local existence and stability for inverse problems to determine a lower-order coefficient and a coefficient of the time derivative involved in a parabolic integro-differential equation from the final data concerning the solution of the direct problem (problems IP2 and IP3). In this connection the previously obtained results for the inverse source problem and the Banach fixed-point theorem can be applied.

In addition to mentioned positivity and monotonicity assumptions, these results require also sufficient smoothness of the data. The stability estimates contain derivatives of the data up to the second order. Therefore, these inverse problems are moderately ill-posed. This complicates the generalization of the results to non-smooth models (e.g. transmission problems). Inverse transmission problems for time-dependent kernels and given additional information along the time axis can be treated assuming additional regularity of the problem in a neighborhood of the trace where the additional information is given [29]. But this is not the case when this information is given in an instant form over the space domain where the direct problem is not regular.

In the second part of the thesis we consider inverse problems to determine parameters in parabolic integro-differential equations using instant and integral additional data in such a manner that the solution is understood in a non-exact sense, namely we deal with quasi-solutions of these problems that minimize certain cost functionals. In this connection we pose the direct problem in a non-regular (weak) form and treat inverse problems to determine several parameters simultaneously.

More precisely, we consider:
• an inverse problem to reconstruct several components of a free term depending either on space or time variables making use of instant conditions given at different time levels (IP4);

• an inverse problem to determine space-dependent components of the free term and an initial condition from integral conditions over time containing different weights (IP5);

• an inverse problem to determine two kernels and a lower order coefficient from a final condition and two conditions for traces of the solution of the direct problem over time (IP6).

First two problems are linear and the latter one is nonlinear. The existence of the quasi-solutions may be proved making use of the Weierstrass existence theorem [65]. The proof is easy for the problems IP4 and IP5, but more complicated for the problem IP6. In latter case we have to show the weak continuity of the solution of the direct problem with respect to the parameters to be recovered. We will do it in the one-dimensional case. In general, the uniqueness of the quasi-solutions may be proved using the strict convexity of the cost functional. Unfortunately, the latter one may not hold for the problems under consideration. However, corresponding regularized problems have unique quasi-solutions due to the strict convexity. The stability issue of quasi-solutions falls outside of the content of the present thesis.

In addition, we prove the Fréchet differentiability of the cost functionals and deduce formulas for the Fréchet derivatives in terms of solutions of certain adjoint problems. To this end we introduce an integrated convolutional form for the weak direct problem. This form does not contain the time derivative of a test function. Operating with such a form of direct problem, we develop a general method to derive adjoint problems and apply it in particular cases.

Finally, we discuss issues related to the gradient method to find the quasi-solutions. The components of the gradient are expressed in terms of the mentioned solutions of the adjoint problems. We will show monotone convergence of the gradient method. This result is a generalization of the former work [18] related to inverse problems for the usual parabolic equation.

The main novelties of the thesis are:

1. a positivity principle for parabolic integro-differential equations is proved;

2. the global existence, uniqueness and stability for an inverse problem to determine a space-dependent component of a free term of a parabolic integro-differential equation in case of given final data are proved;
3. the global uniqueness and local existence and stability for inverse problems to determine a lower-order coefficient and a coefficient of a time derivative of a parabolic integro-differential equation in case of given final data are proved;

4. a general method to derive adjoint problems for Fréchet derivatives of cost functionals corresponding to inverse problems for parabolic integro-differential equations in a weak form is developed and applied particular inverse problems;

5. the existence of quasi-solutions to particular inverse problems with instant and integral additional conditions for parabolic integro-differential equations in a weak form is proved in special cases.

Summing up, the thesis contains a systematical theoretical study of inverse problems for parabolic integro-differential equations with instant and integral conditions, which has not been done before.

The results of the thesis have been presented in the following international meetings:

1. the conference Direct, Inverse and Control Problems for PDE’s - DICOP, Cortona (Italy), 22 - 26.09.2008;


3. 17th International Conference Mathematical Modelling and Analysis, Tallinn, 6-9.06.2012;

4. 18th International Conference Mathematical Modelling and Analysis and 4th International Conference Approximation Methods and Orthogonal Expansions, Tartu, 27-30.05.2013.

Let us give an overview of the contents of the thesis. Thesis contains three chapters.

In Chapter I physical background of the problem is discussed and basic parabolic integrodifferential equation is deduced. Moreover, notation used throughout the thesis is introduced.

Chapter 2 contains results obtained in the smooth case when all terms in the parabolic equation are regular functions. We start by proving basic well-posedness results for the direct problem and establish the positivity principle (§2.1, 2.2). Thereupon, in §2.3 we study the problem to determine the space-dependent component of a free term and in §2.4 we treat the inverse coefficient problems.
Chapter 3 is devoted to the non-smooth case when the parabolic equation contains singular distributions. In §3.1 we prove well-posedness results for the direct problem and introduce the weak convolutional form of the direct problem. Further, in §3.2 we formulate three particular inverse problems in the sense of quasi-solutions, propose the general method to deduce adjoint problems for Fréchet derivatives of cost functionals and apply this method to the posed inverse problems. §3.3 is devoted to the existence of quasi-solutions. In §3.4 briefly the discretization discussed.
1 PHYSICAL BACKGROUND AND NOTATION

1.1 Physical background and the integro-differential equation

Let us deduce the basic parabolic integro-differential equation that will appear in our inverse problems in next chapters.

In linear theory of heat conduction with memory in a medium that is generally inhomogeneous and anisotropic the following constitutive relations are assumed [3, 6, 13, 14, 16, 33, 50, 52, 54]:

\[
q_i(x,t) = -\sum_{j=1}^{n} a_{ij}(x)u_{x_j}(x,t) + \int_{-\infty}^{t} m(t-\tau)\sum_{j=1}^{n} a_{ij}(x)u_{x_j}(x,\tau)d\tau, \\
\]

\[i = 1, \ldots , n,\] (1.1)

\[
e(x,t) = \beta(x)[u(x,t) + \int_{-\infty}^{t} \mu(t-\tau)u(x,\tau)d\tau], \\
\]

where \(x\) is the space variable, \(t\) is the time, \(q = (q_1, \ldots , q_n)\) is the heat flux, \(e\) is the internal energy and \(u\) is the temperature. Moreover, \(a_{ij}\) is the conductivity matrix that in the isotropic case has the form \(a_{ij}(x) = \alpha(x)I\) with some function \(\alpha\) and the unity matrix \(I\), and \(\beta\) is the heat capacity. The functions \(m\) and \(\mu\) are the heat flux relaxation kernel and the internal energy relaxation kernel, respectively. They express the memory of the material. We assume that the memory is synchronous in all points of the medium, i.e. \(m\) and \(\mu\) depend only on the time.

Further, we make use of the continuity equation

\[
e_t(x,t) + \text{div} q(x,t) = \chi(x,t), \\
\]

where \(\chi\) is the source term. Inserting (1.1) and (1.2) into (1.3) and assuming that \(u = 0\) for \(t < 0\) we come to the following parabolic integro-differential equation:

\[
\beta[u + \mu * u]_t = Au - m * Au + \chi, \\
\]

(1.4)

where \(A = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial}{\partial x_j} \right)\).

Here and in the sequel the symbol \(*\) stands for the time convolution, i.e.

\[
v_1 * v_2(t) = \int_{0}^{t} v_1(t-\tau)v_2(\tau)d\tau.
\]

In this thesis we generalize this model mathematically. Namely, we assume the operator \(A\) in Eq. (1.4) be of the form

\[
A = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j} + \sum_{j=1}^{n} a_{j}(x)\frac{\partial}{\partial x_j} + a(x,t), \\
\]

(1.5)
where \(a_{ij}, a_j\) and \(a\) are some coefficients. If necessary, we write \(A(t)\) to indicate the dependence of \(A\) on \(t\).

We will consider the solution \(u\) of the integro-differential equation (1.4) for the arguments 
\[(x,t) \in Q = \Omega \times (0,T),\]
where \(\Omega \in \mathbb{R}^n\) is an \(n\)-dimensional open domain and \(T > 0\) is a fixed number.

We are not going to specify the regularity of \(\Omega\). We simply suppose that \(\Omega\) is sufficiently smooth in order to guarantee our statements to hold. We denote by \(\Gamma\) the boundary of \(\Omega\), by \(\nu(x) = (\nu_1(x), \ldots, \nu_n(x))\) the outer normal of \(\Gamma\) at the point \(x \in \Gamma\) and by \(S\) the boundary cylinder, i.e.
\[S = \Gamma \times (0,T).\]

Throughout the thesis we assume that the \(x\)-dependent coefficient matrix \(a_{ij}\) of the higher order part of the operator \(A\) is uniformly elliptic, i.e.
\[
\sum_{i,j=1}^{n} a_{ij} \lambda_i \lambda_j \geq \epsilon |\lambda|^2 \quad \text{in } \Omega \quad \text{for any } \lambda \in \mathbb{R}^n \quad \text{and some } \epsilon \in (0, \infty) \quad (1.6)
\]
and \(x\)-dependent coefficient \(\beta\) is strictly positive:
\[
\beta \geq \beta_0 \quad \text{in } \Omega \quad \text{with some } \beta_0 \in (0, \infty). \quad (1.7)
\]
(In Sections 2.1 and 2.2 the relations (1.6) and (1.7) will be assumed in a more general form).

### 1.2 Functional spaces

In this section we define most important functional spaces to be used in the study of inverse problems and give some notation.

Firstly, we introduce the Lebesgue spaces of functions defined on a set \(U \subset \mathbb{R}^l\), \(l \in \mathbb{N}\). They are
\[
L^p(U) = \{v : \|v\|_{L^p(U)} = \left[\int_U |v(y)|^p \, dy\right]^{1/p} < \infty\}, \quad 1 \leq p < \infty, \\
L^\infty(U) = \{v : \|v\|_{L^\infty(U)} = \text{ess sup}_{y \in U} |v(y)| < \infty\}.
\]
The space \(L^2(U)\) is a Hilbert space with the inner product \(\langle v, w \rangle_{L^2(U)} = \int_U v(y)w(y) \, dy\). The symbol \(C(U)\) stands for the space of functions, continuous on \(U\). In case \(U\) is compact, \(C(U)\) is a Banach space endowed with the usual maximum norm \(\|v\|_{C(U)} = \max_{y \in U} |v(y)|\).
Next, let $X$ be a Banach space. We generalize the Lebesgue spaces to abstract functions $v$ defined on the interval $(0, T)$ and having values in $X$:

$$L^p(0, T; X) = \{ v : \|v\|_{L^p(0, T; X)} = \left( \int_0^T \|v(t)\|_X^p \, dt \right)^{1/p} < \infty \}, \quad 1 \leq p < \infty,$$

$$L^\infty(0, T; X) = \{ v : \|v\|_{L^\infty(0, T; X)} = \sup_{t \in (0, T)} \|v(t)\|_X < \infty \}$$

and for $1 \leq p \leq \infty$ and $l \in \mathbb{N}$ define the abstract Sobolev spaces:

$$W^l_p(0, T; X) = \left\{ v : \|v\|_{W^l_p(0, T; X)} := \sum_{j=0}^l \|v^{(j)}\|_{L^p(0, T; X)} < \infty \right\}.$$ 

In case $X = \mathbb{R}$, we write merely $W^l_p(0, T; \mathbb{R}) = W^l_p(0, T)$.

Moreover, by $C([0, T]; X)$ we denote the Banach space of abstract functions, continuous on $[0, T]$.

The symbol $\mathcal{L}(X, Y)$ stands for the space of linear bounded operators from a Banach space $X$ to another Banach space $Y$. In case $X = Y$ we write merely $\mathcal{L}(X)$.

In the first part of the thesis (devoted to smooth problems) we need some spaces of fractional order and anisotropic spaces of $x$- and $(x, t)$-dependent real-valued functions. To defined them, let us first introduce the following notation for difference quotients of such functions with powers:

$$\langle v \rangle_p(x_1, x_2) := \frac{v(x_1) - v(x_2)}{|x_1 - x_2|^p}, \quad \langle v \rangle_p(x_1, x_2; t) := \frac{v(x_1, t) - v(x_2, t)}{|x_1 - x_2|^p},$$

$$\langle v \rangle_p(x; t_1, t_2) := \frac{v(x, t_1) - v(x, t_2)}{|t_1 - t_2|^p}.$$

For any real numbers $p \in [1, \infty)$ and $l \in [0, \infty)$ we define the Sobolev-Slobodeckij spaces (cf. [43, 60])

$$W^l_p(\Omega) = \left\{ v : \|v\|_{W^l_p(\Omega)} := \sum_{|\alpha| \leq |l|} \left[ \int_{\Omega} |D^\alpha v(x)|^p \, dx \right]^\frac{1}{p} \right\},$$

$$+ \Theta_l \sum_{|\alpha| = |l|} \left[ \int_{\Omega \times \Omega} |D^\alpha v(x_1, x_2)|^p \, dx_1 \, dx_2 \right]^\frac{1}{p} < \infty,$$

$$W^{l+\frac{1}{2}}_p(Q) = \left\{ v : \|v\|_{W^{l+\frac{1}{2}}_p(Q)} := \sum_{2j + |\alpha| \leq |l|} \left[ \int_{\Omega \times [0, T]} |D^\alpha v(x, t)|^p \, dx \, dt \right]^\frac{1}{p} \right\},$$

$$+ \Theta_l \sum_{2j + |\alpha| = |l|} \left[ \int_{\Omega \times [0, T]} \langle D^\alpha v(x, t) \rangle_p^\frac{1}{p} \, dx \, dt \right]^\frac{1}{p} < \infty,$$

$$+ \Theta_{\frac{l}{2}} \sum_{0 < l - 2j - |\alpha| < 2} \left[ \int_{\Omega \times [0, T]} \langle D^\alpha D x^\alpha v(x, t) \rangle_p^\frac{1}{p} \, dx \, dt \right]^\frac{1}{p} < \infty.$$ 

Here $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is the multi-index with $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $|l|$ is the greatest integer $\leq l$ and $\Theta_l = 0$ and $\Theta_l = 1$ in the cases of integer
and non-integer $l$, respectively. We mention that in the case of integer $l$, $W^l_p(\Omega)$ is the usual Sobolev space of functions defined on $\Omega$.

Furthermore, for any non-integer $l > 0$ we define the Hölder spaces

$$C^l(\Omega) = \left\{ v : D^\alpha v \in C(\Omega) \text{ for } |\alpha| \leq [l] \right\},$$

$$\|v\|_l := \sum_{|\alpha| \leq [l]} \left[ \sup_{x \in \Omega} |D^\alpha v(x)| + \sup_{x_1, x_2 \in \Omega} \|D^\alpha v\|_{l-[\alpha]}(x_1, x_2) \right] < \infty,$$

$$C^{l, \frac{l}{2}}(\Omega) = \left\{ v : D^j_t D^\alpha_x v \in C(\Omega) \text{ for } 2j + |\alpha| \leq [l] \right\},$$

$$\|v\|_{l, \frac{l}{2}} := \sum_{2j + |\alpha| \leq [l]} \left[ \sup_{(x,t) \in \Omega \times [0,T]} |D^j_t D^\alpha_x v(x,t)| + \sup_{(x_1, x_2, t) \in \Omega \times \Omega \times [0,T]} \|D^j_t D^\alpha v\|_{l-[\alpha]}(x_1, x_2; t) \right] < \infty.$$

The definitions of $W^{l, \frac{l}{2}}_p$ and $C^{l, \frac{l}{2}}$ are in a standard manner extended from $Q$ to the boundary cylinder $S$ (for details see [43]). For integer $l \geq 0$ we define

$$C^{2l, l}(\overline{Q}) = \left\{ v : D^j_t D^\alpha_x v \in C(\overline{Q}) \text{ for } 2j + |\alpha| \leq 2l \right\}.$$

Finally, we introduce specific notation related to comparison of real-valued functions. Let $U$ be an open subset or a closer of an open subset in $\mathbb{R}^l$, $l \in \mathbb{N}$, and $f, g : U \rightarrow \mathbb{R}$. We write

$$f \geq g \text{ in } U \quad \text{if } f(x) \geq g(x) \text{ a.e. } x \in U,$$

$$f > g \text{ in } U \quad \text{if for any open set } U_1 \text{ such that } \overline{U}_1 \subseteq U \quad \text{(1.8)}$$

there exists $\varepsilon_{U_1} > 0$ such that $f \geq g + \varepsilon_{U_1}$ in $U_1$. 

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2 SMOOTH PROBLEMS

In this chapter we will pose and study problems for (1.4) under the assumption that this equation holds in the classical sense. The results with some modifications are taken from Publication I.

The starting point is the following initial-boundary value problem

\[
\beta [u + \mu * u]_t = Au - m * Au + \chi \quad \text{in } Q, \quad (2.1)
\]

\[
u = u_0 \quad \text{in } \Omega \times \{0\}, \quad Bu = b \quad \text{in } S, \quad (2.2)
\]

where \( u_0, b \) are given functions, \( B \) is the boundary operator defined either by

\[
Bu = u \quad \text{(we call it case I)} \quad (2.3)
\]

or by

\[
Bu = \omega \cdot \nabla u - m * \omega \cdot \nabla u \quad \text{(we call it case II)}, \quad (2.4)
\]

the operator \( \nabla \) stands for the gradient with respect to the vector of space variables \( x \in \Omega \) and \( \omega(x) = (\omega_1(x), \ldots, \omega_n(x)) \) is an \( x \)-dependent vector satisfying the condition \( \omega \cdot \nu > 0 \). We assume that \( \omega \in (C^1(\Gamma))^n \). We define a number \( \vartheta \) that depends on cases I and II as follows:

\[
\vartheta = \begin{cases} 
0 & \text{in case I} \\
1 & \text{in case II}. 
\end{cases}
\]

This enables unified formulation of statements for direct and inverse problems in both cases (see e.g. Theorem 2.1).

Note that in case II the condition \( Bu = g \) is a generalized boundary condition of the second kind. Indeed, if \( \omega_j = \sum_{i=1}^n a_{ij} \nu_i \) then \( Bu = b \) takes the form \(-q \cdot \nu = b\). This is the physical flux condition.

Let us formulate the following inverse problems that use over-determined final data at \( t = T \) of the solution of (2.1), (2.2) (an inverse free term problem and two inverse coefficient problems).

\textbf{IP1:} Let the free term be of the following form:

\[
\chi(x, t) = z(x)\phi(x, t) + \chi_0(x, t). \quad (2.5)
\]

Given \( \mu, m, \beta, a_{ij}, a_j, a, u_0, b, \phi, \chi_0 \) and a function \( u_T(x), x \in \Omega \), find \( z \) and \( u \) so that the relations (2.1), (2.2), (2.5) and

\[
u = u_T \quad \text{in } \Omega \times \{T\} \quad (2.6)
\]

hold.
Let $a_t = 0$. Given $\mu, m, \beta, a_{ij}, a_j, u_0, b, \chi$ and a function $u_T(x), x \in \Omega$, find $a$ and $u$ so that the relations (2.1), (2.2) and (2.6) hold.

Given $\mu, m, a_{ij}, a_j, a, u_0, b, \chi$ and a function $u_T(x), x \in \Omega$, find $\beta$ and $u$ so that the relations (2.1), (2.2) and (2.6) hold.

It turns out that it is more convenient to treat the direct problem (2.1), (2.2) in case the convolution is removed from the operator $A$. Let us transform (2.1), (2.2) to such a form. Define the resolvent kernel $\hat{m}$ of the kernel $m$ as the solution of the following Volterra integral equation:

$$\hat{m}(t) - \int_0^t m(t - \tau)\hat{m}(\tau)d\tau = m(t), \ t \in (0, T). \quad (2.7)$$

It is well-known that in case $m \in L^p(0, T)$ with $p > 1$ the solution $\hat{m}$ of (2.7) exists, is unique and belongs to $L^p(0, T)$ (see e.g. [15]).

The equality (2.7) implies the following operator relation:

$$(I + \hat{m}*)(I - m*) = I,$$

where $I$ is the unity operator. Bringing the derivative with respect to $t$ into the integral $\mu* u$ and applying the operator $I + \hat{m}$ to the equation (2.1) and the boundary condition (2.2) in case II we transform the relations (2.1), (2.2) to the following form:

$$\beta(u_t + k* u_t) = Au + f \text{ in } Q, \ u = u_0 \text{ in } \Omega \times \{0\}, \ B_1 u = g \text{ in } S, (2.8)$$

where

$$k = \mu + \mu* \hat{m} + \hat{m}, \quad (2.9)$$

$$f = \chi - \beta \mu u_0 + \hat{m}*(\chi - \beta \mu u_0), \quad (2.10)$$

$$B_1 = B, \ g = b \text{ in case I,} \quad (2.11)$$

$$B_1 u = \omega \cdot \nabla u, \ g = b + \hat{m}* b \text{ in case II.} \quad (2.12)$$

Summing up, we can formulate the following lemma.

**Lemma 2.1** In case all $t$-dependent data in (2.1) and (2.2) belong to the space $L^p(0, T)$ with respect to $t$ for any $x$ with some $p > 1$, the problem (2.1), (2.2) is equivalent to the problem (2.8) in a class of functions $u$ such that $u, u_t, u_{x_i}, u_{x_i,x_j}$ belong to $L^p(0, T)$ with respect to $t$ for any $x$.

**2.1 Well-posedness results for direct problem**

In present and the next section we prove existence, uniqueness, stability and a positivity principle for the solution of the direct problem (2.8). These results are used in the study of inverse problems in the subsequent sections.
To the author’s opinion, the positivity principle has a scientific value independently of the inverse problems. Therefore, we try to prove it as generally as we can. For that reason we allow the kernel $k$ and the coefficients $\beta, a_{ij}$ and $a_j$ to depend both on the variables $x$ and $t$ in Sections 2.1 and 2.2. This means that $A$ has the form

$$A = \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} a_j(x,t) \frac{\partial}{\partial x_j} + a(x,t), \quad (2.13)$$

and the basic assumptions (1.6), (1.7) read

$$\sum_{i,j=1}^{n} a_{ij} \lambda_i \lambda_j \geq \epsilon |\lambda|^2 \text{ in } Q \text{ for any } \lambda \in \mathbb{R}^n \text{ and some } \epsilon \in (0, \infty), \quad (2.14)$$

$$\beta \geq \beta_0 \text{ in } Q \text{ with some } \beta_0 \in (0, \infty). \quad (2.15)$$

We start by formulating without a proof a technical lemma. It gives two integral inequalities. Proofs of these inequalities are contained in Publication I, i.e. [27], p. 21 - 23.

**Lemma 2.1** Define $Q_t = \Omega \times (0,t)$, $t > 0$. The following assertions are valid.

(i) Let $k \in L^1(0,T;L^\infty(\Omega))$ and $v \in L^p(Q)$ with some $p \in (1, \infty)$. Then $k * v \in L^p(Q)$ and

$$\|k * v\|_{L^p(Q_t)} \leq \int_0^t \|k(\cdot, t - \tau)\|_{L^\infty(\Omega)} \|v\|_{L^p(Q_\tau)} d\tau, \ t \in (0, T). \quad (2.16)$$

(ii) Let $k \in L^{\frac{2}{l+1}}(0,T;C^l(\Omega))$ and $v \in C^{l, \frac{1}{2}}(Q)$ with some $l \in (0, 1)$. Then $k * v \in C^{l, \frac{1}{2}}(Q)$ and

$$\|k * v\|_{C^{l, \frac{1}{2}}(Q_t)} \leq C_0 \left[ \int_0^t \left\{ \|k(\cdot, t - \tau)\|_l \|v\|_{C^{l, \frac{1}{2}}(Q_\tau)} \right\} \frac{2}{2-l} d\tau \right] \frac{2-l}{2}, \quad (2.17)$$

with some constant $C_0$.

Now let us prove the existence, uniqueness and stability theorem for the direct problem.

**Theorem 2.1** Assume (2.14), (2.15). Then the following assertions are valid.
(i) Let \( \beta, a_{ij}, a_j, a \in C(\Omega) \), \( k \in L^1(0, T; L^\infty(\Omega)) \), \( f \in L^p(\Omega) \), \( u_0 \in W^{2-\frac{2}{p}}_p(\Omega) \) and \( g \in W^{2-\frac{2}{p}-\frac{1}{2p}, \frac{1}{2}-\frac{\alpha}{2}}_{p}(S) \) with some \( p \in (1, \infty) \). Moreover, for case I let \( p \neq \frac{3}{2} \) and the consistency condition \( u_0 = g \) hold in \( \Gamma \times \{0\} \) if \( p > \frac{3}{2} \) and in case II let \( p \neq 3 \) and the consistency condition \( \omega \cdot \nabla u_0 = g \) hold in \( \Gamma \times \{0\} \) if \( p > 3 \). Then the problem (2.8) has a unique solution in the space \( W^{2,1}_p(Q) \). This solution satisfies the estimate

\[
\|u\|_{W^{2,1}_p(Q)} \leq C_1 \left\{ \|f\|_{L^p(\Omega)} + \|u_0\|_{W^{2-\frac{2}{p}}_p(\Omega)} + \|g\|_{W^{2-\frac{\alpha}{2} - \frac{\alpha}{2} - \frac{\alpha}{2}}_{p}(S)} \right\},
\]

(2.18)

where \( C_1 \) is a constant depending on \( \beta, a_{ij}, a_j, a \) and \( k \).

(ii) Let \( \beta, a_{ij}, a_j, a \in C^{l+\frac{1}{2}}(\Omega) \) and \( k \in L^{2l-1}(0, T; C^l(\Omega)) \) with some \( l \in (0, 1) \). Moreover, let \( f \in C^{l+\frac{1}{2}}(\Omega) \), \( u_0 \in C^{2+l}(\Omega) \), \( g \in C^{2+l-\frac{1}{2}l - \frac{\alpha}{2}}(S) \) and in case I the consistency conditions \( u_0 = g \), \( \beta g_t = Au_0 + f \) hold in \( \Gamma \times \{0\} \) and in case II the consistency condition \( \omega \cdot \nabla u_0 = g \) hold in \( \Gamma \times \{0\} \). Then the solution of (2.8) belongs to \( C^{2+l,1+\frac{1}{2}}(Q) \) and satisfies the estimate

\[
\|u\|_{2+l,1+\frac{l}{2}} \leq C_2 \left\{ \|f\|_{l,\frac{l}{2}} + \|u_0\|_{2+l} + \|g\|_{2+l-\frac{1}{2}l - \frac{\alpha}{2} - \frac{\alpha}{2}} \right\},
\]

(2.19)

with some constant \( C_2 \) depending on \( \beta, a_{ij}, a_j, a \) and \( k \).

**Proof.** The assertions (i) and (ii) in the usual parabolic case when \( k = 0 \) were proved in [60]. The proof of the present theorem is based on the Banach fixed point theorem considering the problem (2.8) as a perturbation of the problem in case \( k = 0 \). The contraction is achieved in norms with exponential weights.

Let us start with the assertion (i). By Theorem 5.4 in [60], under the assumptions of (i) problem (2.8) in case \( k = 0 \) has a unique solution \( \hat{u} \) in the space \( W^{2,1}_p(Q) \). Thus, (2.8) for \( u \in W^{2,1}_p(Q) \) in case \( k \neq 0 \) is equivalent to the following problem for the difference \( v = u - \hat{u} \in W^{2,1}_p(Q) \):

\[
\beta v_t = Av - \beta(k * (v_t + \hat{u}_t)) \quad \text{in} \ Q, \quad v = 0 \quad \text{in} \ \Omega \times \{0\}, \quad B_1 v = 0 \quad \text{in} \ S. \tag{2.20}
\]

Let \( F \) stand for the operator that assigns to a function \( \psi \) the solution \( w \) of the problem

\[
\beta w_t = Aw + \psi \quad \text{in} \ Q, \quad w = 0 \quad \text{in} \ \Omega \times \{0\}, \quad B_1 w = 0 \quad \text{in} \ S. \tag{2.21}
\]

By Theorem 5.4 in [60], it holds \( F \in \mathcal{L}(L^p(\Omega), W^{2,1}_p(Q)) \). On the other hand, due to the properties of \( \beta \) and the assertion (i) of Lemma 2.1 we have
\( \beta(k \ast v_t) \in L^p(Q) \) for any \( v \in W^{2,1}_p(Q) \). Consequently, the problem (2.20) in \( W^{2,1}_p(Q_t) \) is equivalent to the following fixed-point equation:

\[
v = Fv + F\hat{u}, \quad \text{where } Fv = F(-\beta(k \ast v_t)) \tag{2.22}
\]

and \( F \in \mathcal{L}(W^{2,1}_p(Q)) \).

Let \( t \in (0, T) \) and define a cutting operator \( P_t \) by the formula

\[
P_t v = \begin{cases} v & \text{in } Q_t \\ 0 & \text{in } Q \setminus Q_t \end{cases}.
\]

Observing that \( Fv = FP_t v \) in \( Q_t \) and using (2.16) we deduce the estimate

\[
\|Fv\|_{W^{2,1}_p(Q_t)} = \|F(\beta(k \ast v_t))\|_{W^{2,1}_p(Q_t)} = \|FP_t(\beta(k \ast v_t))\|_{W^{2,1}_p(Q_t)} \\
\leq \|FP_t(\beta(k \ast v_t))\|_{W^{2,1}_p(Q)} \leq \|F\| \|P_t(\beta(k \ast v_t))\|_{L^p(Q)} \\
= \|F\| \|\beta(k \ast v_t)\|_{L^p(Q)} \leq C_3 \int_0^t \|k(\cdot, t - \tau)\|_{L^\infty(\Omega)} \|v\|_{W^{2,1}_p(Q, \tau)} d\tau
\]

with \( C_3 = \|F\| \|\beta\|_{C(\Omega)} \). Now we define the weighted norms \( \|v\|_\sigma = \sup_{0 < t < T} e^{-\sigma t} \|v\|_{W^{2,1}_p(Q_t)}, \sigma > 0 \), in the space \( W^{2,1}_p(Q) \) and deduce the estimate

\[
\|Fv\|_\sigma \leq C_3 \sup_{0 < t < T} e^{-\sigma t} \int_0^t \|k(\cdot, t - \tau)\|_{L^\infty(\Omega)} \|v\|_{W^{2,1}_p(Q, \tau)} d\tau \tag{2.23}
\]

with \( c_\sigma = C_3 \int_0^T e^{-\sigma \tau} \|k(\cdot, \tau)\|_{L^\infty(\Omega)} d\tau \). Since \( k \in L^1(0, T; L^\infty(\Omega)) \), by the dominated convergence theorem it holds \( c_\sigma \to 0 \) as \( \sigma \to \infty \). Thus, there exists \( \sigma_0 > 0 \) such that \( c_{\sigma_0} < \frac{1}{2} \). Consequently, (2.23) shows that \( F \) is a contraction. We conclude that the equation (2.22), which is equivalent to (2.8), has a unique solution in \( W^{2,1}_p(Q) \).

Further, from (2.22) and (2.23) we deduce

\[
\|v\|_{\sigma_0} \leq \|Fv\|_{\sigma_0} + \|F\hat{u}\|_{\sigma_0} \leq \frac{1}{2}(\|v\|_{\sigma_0} + \|\hat{u}\|_{\sigma_0}).
\]

This implies \( \|v\|_{\sigma_0} \leq \|\hat{u}\|_{\sigma_0} \). Taking into account the equivalency relations of norms \( e^{-\sigma T} \|v\|_0 \leq \|\cdot\|_\sigma \leq \|\cdot\|_0 = \|\cdot\|_{W^{2,1}_p(Q)} \) we further have \( \|v\|_{W^{2,1}_p(Q)} \leq e^{\sigma_0 T} \|\hat{u}\|_{W^{2,1}_p(Q)} \) and by \( u = v + \hat{u} \) we get \( \|u\|_{W^{2,1}_p(Q)} \leq (1 + e^{\sigma_0 T}) \|\hat{u}\|_{W^{2,1}_p(Q)} \).

Recall that \( \tilde{u} \) is the solution of (2.8) in case \( k = 0 \). According to Theorem 5.4 in [60], \( \|\tilde{u}\|_{W^{2,1}_p(Q)} \) is bounded by the right-hand side of (2.18). Consequently, we obtain the estimate (2.18).

Secondly, we prove (ii). Here, in general words, we repeat the part (i) of the proof, but in different spaces. Consider again the above-defined
Due to Theorem 4.9 in [60] it holds \( \hat{u} \in C^{2+l,1+\frac{l}{2}}(Q) \). This implies that the problem (2.8) for \( u \in C^{2+l,1+\frac{l}{2}}(Q) \) is equivalent to the problem (2.20) for the difference \( v = u - \hat{u} \in C^{2+l,1+\frac{l}{2}}(Q) \).

Moreover, recall that \( F \) is the operator that assigns to an element \( \psi \) the solution \( w \) of the problem (2.21). In order to guarantee that \( F\psi \in C^{2+l,1+\frac{l}{2}}(Q) \), the element \( \psi \) must satisfy the consistency condition \( \psi = 0 \) in \( \Gamma \times \{0\} \) in case I. To this end, let us define the following Banach space (that is a subspace of \( C^{l,\frac{1}{2}}(Q) \) in case I):

\[
C^{l,\frac{1}{2}}_0(Q) = \{ \psi \in C^{l,\frac{1}{2}}(Q) : \psi = 0 \text{ in } \Gamma \times \{0\} \text{ in case I} \}
\]

with the norm \( \|\psi\|_{C^{l,\frac{1}{2}}_0(Q)} = \|\psi\|_{C^{l,\frac{1}{2}}(Q)} \).

Due to Theorem 4.9 in [60], again, we have \( F \in \mathcal{L}(C^{l,\frac{1}{2}}_0(Q), C^{2+l,1+\frac{l}{2}}(Q)) \). According to the assumed properties of \( \beta, k \) and the assertion (ii) of Lemma 2.1 we have \( \beta(k \ast v_t) \in C^{l,\frac{1}{2}}(Q) \) for any \( v \in C^{2+l,1+\frac{l}{2}}(Q) \). Moreover, \( \beta(k \ast v_t) \in C^{l,\frac{1}{2}}_0(Q) \) for any \( v \in C^{2+l,1+\frac{l}{2}}(Q) \), because the time convolution is zero for \( t = 0 \). Now we see that the problem (2.8) for \( u \in C^{2+l,1+\frac{l}{2}}(Q) \) is equivalent to the fixed-point equation (2.22) for \( v = u - \hat{u} \in C^{2+l,1+\frac{l}{2}}(Q) \), where \( F \in \mathcal{L}(C^{2+l,1+\frac{l}{2}}(Q)) \).

Let \( t \in (0, T) \) and define a continuation operator \( \tilde{P}_t \) by the formula

\[
\tilde{P}_t v(x, \tau) = \begin{cases} 
 v(x, \tau) & \text{for } (x, \tau) \in Q_t \\
 v(x, t) & \text{for } (x, \tau) \in Q \setminus Q_t 
\end{cases}
\]

Then, it hold \( \|\tilde{P}_t v\|_{l,\frac{1}{2}} = \|v\|_{C^{l,\frac{1}{2}}(Q_t)} \). and \( Fv = F\tilde{P}_t v \) in \( Q_t \). Thus, using (2.17) we obtain

\[
\|Fv\|_{C^{2+l,1+\frac{l}{2}}(Q_t)} = \|F(\beta(k \ast v_t))\|_{C^{2+l,1+\frac{l}{2}}(Q_t)} = \|F\tilde{P}_t(\beta(k \ast v_t))\|_{C^{2+l,1+\frac{l}{2}}(Q_t)} \leq \|F\tilde{P}_t(\beta(k \ast v_t))\|_{2+l,1+\frac{l}{2}} \leq \|F\| \|\beta(k \ast v_t)\|_{C^{l,\frac{1}{2}}(Q_t)} \leq C_4 \left[ \int_0^t \left\{ \|k(\cdot, t - \tau)\|_{l,\frac{1}{2}} \|v\|_{C^{2+l,1+\frac{l}{2}}(Q, \tau)} \right\}^{\frac{1}{2-l}} d\tau \right]^{\frac{2-l}{2}} ,
\]

where \( C_4 = C_0\|F\|\|\beta\|_{l,\frac{1}{2}} \). Defining the weighted norms \( \|w\|_\sigma \)
\[
\sup_{0 < t < T} e^{-\sigma t} \|v\|_{C^{2+l,1+\frac{l}{2}}(Q_t)}, \quad \sigma > 0,
\]

in the space \(C^{2+l,1+\frac{l}{2}}(Q_t)\), we deduce

\[
\begin{align*}
\|Fv\|_{\sigma} & \leq C_4 \sup_{0 < t < T} e^{-\sigma t} \left[ \int_0^t \left\{ \|k(\cdot, t - \tau)\|_{l} \|v\|_{C^{2+l,1+\frac{l}{2}}(Q_\tau)} \right\}^{\frac{2}{2-l}} d\tau \right]^{\frac{2-l}{2}} \\
& = C_4 \sup_{0 < t < T} \left[ \int_0^t \left\{ e^{-\sigma(t-\tau)} \|k(\cdot, t - \tau)\|_{l} \times e^{-\sigma \tau} \|v\|_{C^{2+l,1+\frac{l}{2}}(Q_\tau)} \right\}^{\frac{2}{2-l}} d\tau \right]^{\frac{2-l}{2}} \\
& \leq C_4 \left[ \int_0^T \left\{ e^{-\sigma \tau} \|k(\cdot, t - \tau)\|_{l} \right\}^{\frac{2}{2-l}} d\tau \right]^{\frac{2-l}{2}} \|v\|_{\sigma}.
\end{align*}
\]

Since \(k \in L_{2-l}^2(0, T; C^l(\Omega))\), by the dominated convergence theorem, the coefficient \(\left[ \int_0^T \left\{ e^{-\sigma \tau} \|k(\cdot, t - \tau)\|_{l} \right\}^{\frac{2}{2-l}} d\tau \right]^{\frac{2-l}{2}}\) is small for large \(\sigma\). Owing to this, the proof can be finished as in case (i) making use of the fixed-point argument and an estimate for \(\|\hat{u}\|_{2+l,1+\frac{l}{2}}\) in Theorem 4.9 of [60].

### 2.2 Positivity principle

In this section we prove a positivity principle for the solution of (2.8).

**Theorem 2.2** Assume (2.14), (2.15), \(k \in W_1^1(0, T; L^\infty(\Omega))\), \(\beta, a_{ij}, a_j, a \in C(\overline{Q})\)

\[
k \geq 0, \quad k_t \leq 0.
\]

Let \(u \in W_0^{2,1}(Q)\) with some \(p \in (1, \infty)\) solve the problem (2.8) and \(u_0 \geq 0, \quad g \geq 0, \quad f \geq 0\). Then the following assertions are valid:

1. \(u \geq 0;\)

2. if, in addition, \(\beta, a_{ij}, a_j, a \in C^{1,\frac{l}{2}}(Q)\) with some \(l \in (0, 1)\) and there exists an open subset \(Q_f\) of \(Q\) such that \(f > 0\) in \(Q_f\), then \(u(\cdot, T) > 0\) in \(\Omega\) in case I and \(u(\cdot, T) > 0\) in \(\overline{\Omega}\) in case II.

**Proof.** It consists of 4 steps.

1. **step.** We prove the assertion (i) under the additional assumptions

\[
u \in C^{2,1}(\overline{Q}), \quad k \in W_1^1(0, T; C(\overline{\Omega})), \quad a \leq 0.
\]

Since \(u = u_0 \geq 0\) in \(\Omega \times \{0\}\), there exists

\[
t_0 = \sup \{ t : u(x, \tau) \geq 0 \text{ for } (x, \tau) \in \Omega \times [0, t], \ 0 \leq t \leq T \}.
\]
In case the assertion \( u \geq 0 \) holds in \( Q \), we have \( t_0 = T \). Suppose on the contrary that \( t_0 < T \). Then, we fix some \( h \in (0, T - t_0] \) and define the set \( V_{t_0,h} = \overline{\Omega} \times (t_0, t_0+h] \). Note that the closure of \( V_{t_0,h} \) is \( V_{t_0,\infty} = \overline{\Omega} \times [t_0, \infty) \). By the definition of \( t_0 \) and \( V_{t_0,h} \), there exists \( (x_{h}^{*}, t_{h}^{*}) \in V_{t_0,h} \) such that \( u(x_{h}^{*}, t_{h}^{*}) < 0 \). Let us introduce the following function:

\[
v(x, t) = u(x, t) + \mu_h (t - t_0 - h), \quad \text{where} \quad \mu_h = -\frac{u(x_{h}^{*}, t_{h}^{*})}{2h} > 0. \tag{2.27}
\]

Since \(-\mu_h h \leq \mu_h (t - t_0 - h) \leq 0 \) for \( t \in [t_0, t_0 + h] \), we have

\[
u(x, t) - \mu_h h \leq v(x, t) \leq u(x, t) \quad \text{for} \quad (x, t) \in \overline{V_{t_0,h}}. \tag{2.28}
\]

Observing (2.28), the definition of \( \mu_h \) and the inequality \( u(x_{h}^{*}, t_{h}^{*}) < 0 \), we see that for all \((x, t) \in \overline{V_{t_0,h}}\) such that \( u(x, t) \geq 0 \), the relations

\[
v(x, t) \geq u(x, t) - \mu_h h \geq -\mu_h h = \frac{u(x_{h}^{*}, t_{h}^{*})}{2} > u(x_{h}^{*}, t_{h}^{*}) \geq v(x_{h}^{*}, t_{h}^{*})
\]

are valid. They imply that

function \( v \) cannot attain its minimum over \( \overline{V_{t_0,h}} \)

in a point \((x, t)\) where \( u(x, t) \geq 0 \). \tag{2.29}

In particular, (2.29) implies that \( v \) cannot attain its minimum over \( \overline{V_{t_0,h}} \) on the subset \( \overline{V_{t_0,h}} \setminus V_{t_0,h} = \overline{\Omega} \times \{t_0\} \), because there \( u \geq 0 \) in view of the definition of \( t_0 \). Therefore,

\[
\exists (x_h, t_h) \in V_{t_0,h} : \quad v(x_h, t_h) \leq v(x, t) \quad \text{for all} \quad (x, t) \in \overline{V_{t_0,h}}.
\]

Moreover, \( v(x_h, t_h) < 0 \), because \( v(x_h, t_h) \leq v(x_{h}^{*}, t_{h}^{*}) \leq u(x_{h}^{*}, t_{h}^{*}) \) and \( u(x_{h}^{*}, t_{h}^{*}) < 0 \).

Let us show that \( x = x_h \) is the stationary minimum point of the \( x \)-dependent function \( v(x, t_h) \), i.e.

\[
\nabla v(x_h, t_h) = 0. \tag{2.30}
\]

This relation may fail only in case the minimum occurs in the lateral boundary of \( V_{t_0,h} \), i.e. when \( x_h \in \Gamma \). In case I we have \( u = g \geq 0 \) for \( x \in \Gamma \) and, by statement (2.29), \( x_h \) cannot belong to \( \Gamma \). Thus, it remains to show (2.30) for the case II when \( x_h \in \Gamma \). In this case \( \omega \cdot \nabla v = \omega \cdot \nabla u = g \) in \( \Gamma \). Note that then the inequality \( g(x_h, t_h) > 0 \) cannot hold, because otherwise \( v \) is strictly decreasing in the inner direction \( -\omega(x_h) \) at \((x_h, t_h)\) which implies that \( v(x_h, t_h) \) cannot be the minimum of \( v \). Consequently, due to the assumption \( g \geq 0 \), it holds \( g(x_h, t_h) = 0 \) and we have \( \omega(x_h) \cdot \nabla v(x_h, t_h) = 0 \).

In addition, in case \( n \geq 2 \) we also have \( \tau \cdot \nabla v(x_h, t_h) = 0 \), where \( \tau \) is an
arbitrary tangential direction of $\Gamma$ at $x_h$, because $x = x_h$ is the minimum point of the $x$-dependent function $v(x, t_h)$ over the set $\Gamma$. Summing up, $\xi \cdot \nabla v(x_h, t_h) = 0$, where $\xi$ is any direction. We obtain (2.30).

Now we are going to estimate the operator $Lu = \beta(u_t + k \ast u_t) - Au$ of the equation (2.8) termwise at the point $(x, t) = (x_h, t_h)$. By (2.27) we have $u_t(x, t_h) = v_t(x, t_h) - \mu_h$. Since $t_h$ is the minimum point of the $t$-dependent function $v(x, t)$ in the half-interval $(t_0, t_0 + h]$, it holds $v_t(x_h, t_h) \leq 0$. Thus, we obtain

$$u_t(x_h, t_h) \leq -\mu_h. \quad (2.31)$$

Substituting $u$ by $v - \mu_h(t - t_0 - h)$ in the right-hand side of (2.13) we have $Au = \sum_{i,j=1}^{N} a_{ij}v_{x_i x_j} + \sum_{j=1}^{n} a_j v_{x_j} + a[v - \mu_h(t - t_0 - h)]$. Since $x = x_h$ is the stationary minimum point of $v(x, t_h)$ and the principal part of $A$ is elliptic (see (2.14)), the relations $\sum_{j=1}^{n} a_j v_{x_j}$ and $\sum_{i,j=1}^{n} a_{ij}v_{x_i x_j} \geq 0$ are valid at $(x, t) = (x_h, t_h)$. Thus, $-Au(x_h, t_h) \leq a(x_h, t_h)[\mu_h(t_h - t_0 - h) - v(x_h, t_h)]$. By the additional assumption $a \leq 0$ and and the inequality $v(x_h, t_h) < 0$ we further get $-Au(x_h, t_h) \leq a(x_h, t_h)\mu_h(t_h - t_0 - h)$. Since $|t_h - t_0 - h| \leq h$ we deduce the estimate

$$-Au(x_h, t_h) \leq C_5 \mu_h h, \quad (2.32)$$

where $C_5 = \|a\|_{C(\overline{Q})}$.

Finally, we estimate the term $(k \ast u_t)(x_h, t_h)$ in $Lu(x_h, t_h)$. Integrating by parts we have

$$\int_{0}^{t_h} k(x_h, t_h - \tau)u_\tau(x_h, \tau)d\tau = k(x_h, 0)u(x_h, t_h) - k(x_h, t_h)u_0(x_h)$$

$$+ \int_{0}^{t_0} k_t(x_h, t_h - \tau)u(x_h, \tau)d\tau + \int_{t_0}^{t_h} k_t(x_h, t_h - \tau)u(x_h, \tau)d\tau.$$

Here $-k(x_h, t_h)u_0(x_h) \leq 0$ and $\int_{0}^{t_0} k_t(x_h, t_h - \tau)u(x_h, \tau)d\tau \leq 0$, because $k \geq 0$, $k_t \leq 0$, $u_0 \geq 0$ and $u(x_h, \tau) \geq 0$ for $\tau \in [0, t_0]$ by the definition of $t_0$. Consequently, we can estimate as follows:

$$\int_{0}^{t_h} k(x_h, t_h - \tau)u_\tau(x_h, \tau)d\tau \leq k(x_h, 0)u(x_h, t_h) + \int_{t_0}^{t_h} k_t(x_h, t_h - \tau)u(x_h, \tau)d\tau.$$
Substituting in the right-hand side \( u(x_h, t) \) by \( v(x_h, t) - \mu_h(t - t_0 - h) \), we get

\[
\int_0^{t_h} k(x_h, t_h - \tau) u_{\tau}(x_h, \tau) d\tau
\]

\[
\leq k(x_h, 0)v(x_h, t_h) + \int_{t_0}^{t_h} k_{t}(x_h, t_h - \tau)v(x_h, \tau)d\tau -
\]

\[\mu_h \left[ k(x_h, 0)(t_h - t_0 - h) + \int_{t_0}^{t_h} k_{t}(x_h, t_h - \tau)(\tau - t_0 - h)d\tau \right].\]

In this relation we analyze separately the term \( \int_{t_0}^{t_h} k_{t}(x_h, t_h - \tau)v(x_h, \tau)d\tau \).

To this end, introduce the following subsets of \([t_0, t_h]\):

\[
U^+_h = \{ \tau \in [t_0, t_h] : v(x_h, \tau) \geq 0 \}, \quad U^-_h = \{ \tau \in [t_0, t_h] : v(x_h, \tau) < 0 \}.
\]

Taking account of \( k_t \leq 0 \) and the fact that \( v(x_h, t_h) < 0 \) is the minimum of \( v(x_h, t) \) on the interval \([t_0, t_h]\), we deduce

\[
\int_{t_0}^{t_h} k_{t}(x_h, t_h - \tau)v(x_h, \tau)d\tau
\]

\[= \int_{U^+_h} k_{t}(x_h, t_h - \tau)v(x_h, \tau)d\tau + \int_{U^-_h} k_{t}(x_h, t_h - \tau)v(x_h, \tau)d\tau \leq \int_{U^-_h} k_{t}(x_h, t_h - \tau)v(x_h, \tau)d\tau \cdot v(x_h, t_h) \]

\[\leq \int_{t_0}^{t_h} k_{t}(x_h, t_h - \tau)d\tau \cdot v(x_h, t_h) = (k(x_h, t_h - t_0) - k(x_h, 0)) v(x_h, t_h).\]

Using this estimate in (2.33) and taking into account \( k \geq 0, v(x_h, t_h) < 0 \) we obtain

\[
\int_0^{t_h} k(x_h, t_h - \tau) u_{\tau}(x_h, \tau) d\tau \leq k(x_h, 0)v(x_h, t_h)
\]

\[+ (k(x_h, t_h - t_0) - k(x_h, 0)) v(x_h, t_h)
\]

\[\mu_h \left[ k(x_h, 0)(t_h - t_0 - h) + \int_{t_0}^{t_h} k_{t}(x_h, t_h - \tau)(\tau - t_0 - h)d\tau \right]
\]

\[\leq -\mu_h \left[ k(x_h, 0)(t_h - t_0 - h) + \int_{t_0}^{t_h} k_{t}(x_h, t_h - \tau)(\tau - t_0 - h)d\tau \right].\]

(2.34)

Let us put all pieces together. Making use of (2.31), (2.32), (2.34), the assumption (2.15) and the relations \( \mu_h > 0 \) and \( |\tau - t_0 - h| \leq h \) for
$\tau \in [t_0, t_h]$ we obtain

\[
Lu(x_h, t_h) \leq \mu_h \left\{ -\beta(x_h, t_h) - \beta(x_h, t_h) [k(x_h, 0)(t_h - t_0 - h) + \int_{t_0}^{t_h} k_t(x_h, t_h - \tau)(\tau - t_0 - h)d\tau] + C_5 h \right\} \\
\leq \mu_h \left\{ -\beta_0 + h \left( \|\beta\|_{C(\overline{Q})} [k(x_h, 0) + \|k_t\|_{L^1(0, T; C(\overline{Q}))}] + C_5 \right) \right\}.
\]

In case $h > 0$ is sufficiently small, due to the inequalities $\mu_h > 0$ and $\beta_0 > 0$ the relation $Lu(x_h, t_h) < 0$ holds. But this contradicts to the assumption $Lu = f \geq 0$. Consequently, the supposition $t_0 < T$ was not right. It holds $t_0 = T$, which by the definition of $t_0$ implies $u \geq 0$.

2. step. We prove the assertion (i) under the additional assumptions

\[
u \in C^{2,1}(\overline{Q}), \; k \in W^1_1(0, T; C(\overline{Q})), \; u_0 = 0. \tag{2.35}
\]

Let us define $\tilde{u} = e^{-\sigma t}u$, where $\sigma = \frac{\|a\|_{C(\overline{Q})}}{\beta_0}$ and insert $u = e^{\sigma t}\tilde{u}$ to the equation (2.8). Expressing the $t$-derivative as follows: $u_t = e^{\sigma t}\tilde{u}_t + \sigma e^{\sigma t}\tilde{u}$ and dividing the equation by $e^{\sigma t}$ we obtain

\[
\beta \left( \tilde{u}_t + \sigma \tilde{u} + e^{\sigma t}k \ast \left( e^{\sigma t}\tilde{u}_t + \sigma e^{\sigma t}\tilde{u} \right) \right) = A\tilde{u} + e^{\sigma t}f. \tag{2.36}
\]

Let us transform the term with $k$ in this equation:

\[
e^{-\sigma t}k \ast \left( e^{\sigma t}\tilde{u}_t + \sigma e^{\sigma t}\tilde{u} \right) \\
= \int_0^t e^{-\sigma(t-\tau)}k(x, t - \tau)\tilde{u}_\tau(x, \tau)d\tau + \sigma \int_0^t e^{-\sigma(t-\tau)}k(x, t - \tau)\tilde{u}(x, \tau)d\tau \\
= \int_0^t e^{-\sigma(t-\tau)}k(x, t - \tau)\tilde{u}_\tau(x, \tau)d\tau + \sigma \int_0^t \int_0^\tau e^{-\sigma\eta}k(x, \eta)d\eta\tilde{u}_\tau(x, \tau)d\tau \\
= k * \tilde{u}_t,
\]

where $\tilde{k}(x, t) = e^{-\sigma t}k(x, t) + \sigma \int_0^t e^{-\sigma\eta}k(x, \eta)d\eta$. Here we used the relation $\tilde{u}(\cdot, 0) = u_0 = 0$ during the integration by parts.

Thus, the equation (2.36) takes the form $\beta(\tilde{u}_t + \tilde{k} * \tilde{u}_t) = \tilde{A}\tilde{u} + \tilde{f}$, where

$\tilde{f} = e^{\sigma t}f$, $\tilde{A}\tilde{u} = \sum_{i,j=1}^n a_{ij}\tilde{u}_{x_ix_j} + \sum_{j=1}^n a_j\tilde{u}_{x_j} + \tilde{a}\tilde{u}$ and $\tilde{a} = a - \sigma \beta$.

Summing up, $\tilde{u}$ solves the problem

$\beta(\tilde{u}_t + \tilde{k} * \tilde{u}_t) = \tilde{A}\tilde{u} + \tilde{f}$ in $Q$, $\tilde{u} = 0$ in $\Omega \times \{0\}$, $B_1\tilde{u} = \tilde{g}$ in $S$,

where $\tilde{g} = e^{-\sigma t}g$. Since $k \in W^1_1(0, T; C(\overline{Q}))$, it holds $\tilde{k} \in W^1_1(0, T; C(\overline{Q}))$. The inequalities (2.25) yield $\tilde{k} \geq 0$, $\tilde{k}_t \leq 0$. Moreover, $\tilde{f} \geq 0$, $\tilde{g} \geq 0$, and
due to the choice of $\sigma$ and the condition (2.15), it holds $\tilde{a} \leq 0$. By means of the part 1 of the proof, we get $\tilde{u} \geq 0$. This implies $u \geq 0$.

3. step. We prove the assertion (i) in the general case. It is enough to prove this assertion for $p \in (1, \frac{3}{2})$. Indeed, a solution $u$ of (2.8) that belongs to the space $W_{p}^{2,1}(Q)$ for some $p \in [\frac{3}{2}, \infty)$ belongs to such a space for any $p \in (1, \frac{3}{2})$, too. Operating with solutions in $W_{p}^{2,1}(Q)$, $p \in (1, \frac{3}{2})$, we have not to deal with consistency conditions.

Let us formulate two problems:

$$\beta \tilde{u}_t = (A - k(\cdot, 0)\beta)\tilde{u} \text{ in } Q, \quad \tilde{u} = u_0 \text{ in } \Omega \times \{0\}, \quad B_1 \tilde{u} = g \text{ in } S, \quad (2.37)$$

$$\beta(\tilde{u}_t + k \ast \tilde{u}_t) = A\tilde{u} + f_\tilde{u} \text{ in } Q, \quad \tilde{u} = 0 \text{ in } \Omega \times \{0\}, \quad B_1 \tilde{u} = 0 \text{ in } S, \quad (2.38)$$

where $f_\tilde{u} = f - \beta(k_t \ast \tilde{u}) + \beta ku_0$.

Firstly, we establish the existence of solutions of these problems. Due to $u \in W_{p}^{2,1}(Q)$ and embedding theorems [43] we have $u_0 = u|_{\Omega \times \{0\}} \in W_{p}^{2-\frac{2}{p}}(\Omega)$ and $g = B_1u|_{S} \in W_{p}^{2-\frac{1}{p}-\frac{1}{2p} - \frac{\tilde{u}}{2}}(S)$. Since $k \in W_{\infty}^{1}(0, T; L^{\infty}(\Omega))$ and $\beta \in C(\bar{Q})$, the coefficient of the lower order term $k(\cdot, 0)\beta$ of the parabolic equation in (2.37) belongs to $L^{\infty}(Q)$. Using Theorem 5.4 in [60] we conclude that problem (2.37) has a unique solution $\bar{u} \in W_{p}^{2,1}(Q)$. Further, due to $\beta \in C(\bar{Q})$, $k \in W_{\infty}^{1}(0, T; L^{\infty}(\Omega)) \subset L^{\infty}(Q)$ and $u_0 \in W_{p}^{2-\frac{2}{p}}(\Omega) \subset L^{p}(\Omega)$ we have $\beta ku_0 \in L^{p}(Q)$. Moreover, in view of $k_t \in L^{1}(0, T; L^{\infty}(\Omega))$, $\bar{u} \in W_{p}^{2,1}(Q) \subset L^{p}(Q)$ and the Young’s theorem for convolutions, we get $k_t \ast \bar{u} \in L^{p}(Q)$, which implies $\beta(k_t \ast \bar{u}) \in L^{p}(Q)$. Consequently, due to $f \in L^{p}(Q)$ we obtain $f_\bar{u} \in L^{p}(Q)$. By Theorem 2.1 (i), problem (2.38) has a unique solution $\bar{u} \in W_{p}^{2,1}(Q)$.

Adding the problems (2.37) and (2.38) and integrating by parts the convolution term in $f_\bar{u}$, we see that the sum $\bar{u} + \tilde{u}$ solves (2.8). Hence, by the uniqueness, it holds $u = \bar{u} + \tilde{u}$. According to the well-known extremum principle for parabolic equations (e.g. [43] Ch. 3, Theorem 7.2), we have $\tilde{u} \geq 0$. This together with the assumptions of theorem implies $f_\tilde{u} \geq 0$. In order to complete the step 3 it remains to show that $\bar{u} \geq 0$.

The idea to prove $\bar{u} \geq 0$ consists in approximation of the problem (2.38) by a sequence of smooth problems and applying the result of step 2 to latter ones. Let us choose some functions $\beta^m, a_{ij}^m, a_i^m, a^m, \hat{f}^m, \hat{k}^m \in C^{\infty}(\bar{Q})$ such that

$$\|\beta^m - \bar{\beta}\|_{C(\bar{Q})}, \|a_{ij}^m - a_{ij}\|_{C(\bar{Q})}, \|a_i^m - a_i\|_{C(\bar{Q})}, \|a^m - a\|_{C(\bar{Q})}, \|\hat{f}^m - f_\bar{u}\|_{L^p(Q)}, \|\hat{k}^m - k\|_{W_{1}^{1}(0, T; L^{\infty}(\Omega))} \to 0 \quad \text{as} \quad m \to \infty \quad (2.39)$$

and $\hat{f}^m = 0$ in $\Gamma \times \{0\}$. Due to (2.39), for sufficiently large $m \geq M_{\epsilon, \beta_0}$ the relations $\max_{k=1}^{\hat{k}^m} \|a_{ij}^m - a_{ij}\|_{C(\bar{Q})} \leq \frac{\epsilon}{2m}$ and $\|\beta^m - \bar{\beta}\|_{C(\bar{Q})} \leq \frac{\beta_0}{2}$ are valid. Therefore, from (2.14) and (2.15) we get the following ellipticity and positivity
inequalities for the matrix $a_{ij}^m$ and the coefficient $\beta^m$ in case $m \geq M_{\epsilon, \beta_0}$:

$$\sum_{i,j=1}^n a_{ij}^m(x, t) \lambda_i \lambda_j \geq \frac{\epsilon}{2} |\lambda|^2 \quad \text{for any } x \in \Omega, \ t \in (0, T), \ \lambda \in \mathbb{R}^n,$$

$$\beta^m(x, t) \geq \frac{\beta_0}{2} \quad \text{for any } x \in \Omega, \ t \in (0, T).$$

Further, on the basis of $\hat{f}^m$, let us define the nonnegative approximation $f^m(x, t) = \frac{|\hat{f}^m(x, t)| + \hat{f}^m(x, t)}{2}$ for the function $f$. Due to $f \geq 0$ the inequality $|f^m - f| \leq |\hat{f}^m - f|$ holds. Therefore, in view of (2.39) we obtain

$$\|f^m - f\|_{L^p(Q)} \leq \|\hat{f}^m - f\|_{L^p(Q)} \to 0 \quad \text{as } m \to \infty. \quad (2.40)$$

The operation of taking the absolute value preserves the Hölder-continuity of a function. Therefore, $f^m \in C^{l, \frac{\delta}{2}}(Q)$ for any $l \in (0, 1)$. The relation $f^m = 0$ in $\Gamma \times \{0\}$ yields $f^m = 0$ in $\Gamma \times \{0\}$.

Finally, on the basis of $\hat{k}^m$ we define the new approximation for $k$:

$$k^m(x, t) = \int_T^t \tau^m(x, \tau) d\tau + q^m(x),$$

where $\tau^m(x, t) = \frac{\hat{k}^m(x, t) - |\hat{k}^m(x, t)|}{2}$ and $q^m(x) = \frac{|\hat{k}^m(x, T)| + \hat{k}^m(x, T)}{2}$. Then $k^m \in W_1^l(0, T; C^l(\Omega))$ for any $l \in (0, 1)$ and $k^m \geq 0$, $k_{t}^m \leq 0$. Moreover, since $k_t \leq 0$ and $k \geq 0$, we obtain $|\tau^m - k_t| \leq |\hat{k}_{t}^m - k_t|$ and $|q^m - k(\cdot, T)| \leq |\hat{k}^m(\cdot, T) - k(\cdot, T)|$. Observing these inequalities, the relation $k(x, t) = \int_T^t (\hat{k}(x, T) - k(x, \tau)) d\tau + k(x, T)$ and (2.39) we deduce

$$\|k^m - k\|_{L^1(0, T; L^\infty(\Omega))} \leq \int_0^T \text{ess sup}_{x \in \Omega} \left[ \int_t^T |\tau^m(x, \tau) - k(x, \tau)| d\tau + |q^m(x) - k(x, T)| \right] dt$$

$$\leq T \left[ \|\hat{k}_{t}^m - k_t\|_{L^1(0, T; L^\infty(\Omega))} + \|\hat{k}^m(\cdot, T) - k(\cdot, T)\|_{L^\infty(\Omega)} \right] \to 0$$

as $m \to \infty. \quad (2.41)$

Now let us formulate the following sequence of approximating problems for (2.38):

$$\begin{align*}
\beta^m(u^m_t + k^m * u^m) &= A^m u^m + f^m \quad \text{in } Q, \\
u^m &= 0 \quad \text{in } \Omega \times \{0\}, \quad B_1 u^m = 0 \quad \text{in } S,
\end{align*} \quad (2.42)$$

where $A^m = \sum_{i,j=1}^n a_{ij} v_{x_i} v_{x_j} + \sum_{j=1}^n a_j^m v_{x_j} + a^m v$. Observing the proved regularity of the data of these problems and Theorem 2.1 (ii) we conclude that for each integer $m \geq M_{\epsilon, \beta_0}$, (2.42) has the unique solution $u^m \in C^{2+l, 1+\frac{\delta}{2}}(Q)$. 31
Subtracting (2.38) from (2.42) we obtain the following problems for the differences \( v^m = u^m - \tilde{u} \):
\[
\beta(v_t^m + k * v_t^m) = A v^m + \phi^m \text{ in } Q, \ v^m = 0 \text{ in } \Omega \times \{0\}, \ B_1 v^m = 0 \text{ in } S
\]
where
\[
\phi^m = (A^m - A)(v^m + \tilde{u}) - (\beta^m - \beta)[v_t^m + \tilde{u}_t + k * (v_t^m + \tilde{u}_t)]
- (\beta + \beta^m - \beta)(k^m - k) * (v_t^m + \tilde{u}_t) + f^m - f_{\tilde{u}}.
\]
Using the estimate (2.18) and the Young’s inequality for convolutions we deduce
\[
\|v^m\|_{W^{2,1}_p(Q)} \leq C_1 \|\phi^m\|_{L^p(Q)} \leq C_1 \|f^m - f_{\tilde{u}}\|_{L^p(Q)}
+ \tilde{C}(m)(\|v^m\|_{W^{2,1}_p(Q)} + \|\tilde{u}\|_{W^{2,1}_p(Q)}),
\]
where
\[
\tilde{C}(m) = C_6 \left[ \|a_{ij}^m - a_{ij}\|_{(C(\overline{Q}))^{n \times n}} + \|a_j^m - a_j\|_{(C(\overline{Q}))^n} + \|a^m - a\|_{C(\overline{Q})}
+ \|\beta^m - \beta\|_{C(\overline{Q})} + \|\beta^m - \beta\|_{C(\overline{Q})} \|k^m - k\|_{L^1(0, T; L^\infty(\Omega))} \right]
\]
and \( C_6 \) is a constant independent of \( m \). By virtue of (2.39) and (2.41) we have \( \tilde{C}(m) \to 0 \) as \( m \to \infty \). Hence, there exists \( \tilde{M} \geq M_{e, \beta_0} \) such that \( \tilde{C}(m) \leq \frac{1}{2} \) for \( m \geq \tilde{M} \). We get \( \|v^m\|_{W^{2,1}_p(Q)} \leq 2C_1 \|f^m - f_{\tilde{u}}\|_{L^p(Q)} + 2\tilde{C}(m)\|\tilde{u}\|_{W^{2,1}_p(Q)} \) for \( m \geq \tilde{M} \). This in view of (2.40) implies
\[
\|v^m\|_{W^{2,1}_p(Q)} = \|u^m - \tilde{u}\|_{W^{2,1}_p(Q)} \to 0 \text{ as } m \to \infty. \tag{2.43}
\]
Recall that \( f^m \geq 0 \), \( k^m \geq 0 \) and \( k_t^m \leq 0 \). Thus, according to step 2 of the proof, the solution of the problem (2.42) satisfies \( u^m \geq 0 \). This with (2.43) implies \( \tilde{u} \geq 0 \). This yields \( u \geq 0 \). Step 3 is completed. The assertion (i) of the theorem is proved.

4. step. We prove (ii). According to (1.8), there exists an open ball \( U \subset Q_f \) and \( \varepsilon > 0 \) such that \( f \geq \varepsilon \) in \( U \). We can choose some \( f^\dagger \in C^\infty(Q) \) so that \( f^\dagger = 0 \) in \( Q \setminus U \) and \( 0 < f^\dagger \leq \frac{\varepsilon}{2} \) in \( U \). (For instance, we can set
\[
f^\dagger(y) = \frac{\varepsilon}{2} e^{-\frac{|y-y_0|^2}{\rho^2}} \text{ in } U, \text{ where } y = (x, t) \text{ and } y_0 \text{ and } \rho \text{ are the center and the radius of the ball } U, \text{ respectively.}) \text{ Then } f^\dagger \leq f \text{ in } Q.
\]

Further, let us define \( c_k = \text{ess sup}_{x \in \Omega} k(x, 0) \) and formulate the following problems:
\[
\beta u_t^\dagger = (A - c_k \beta) u^\dagger + f^\dagger \text{ in } Q, \ u^\dagger = 0 \text{ in } \Omega \times \{0\}, \ B_1 u^\dagger = 0 \text{ in } S, \tag{2.44}
\]
\[
\beta(\tilde{u}_t + k * \tilde{u}_t) = A \tilde{u} + \tilde{f}_u^\dagger \text{ in } Q, \ \tilde{u} = u_0 \text{ in } \Omega \times \{0\}, \ B_1 \tilde{u} = g \text{ in } S, \tag{2.45}
\]
where \( \tilde{f}_u^\dagger = f - f^\dagger - \beta(k_t * u^\dagger) + \beta(c_k - k(\cdot, 0)) u^\dagger \). By Theorem 2.1 (ii), problem (2.44) has a unique solution \( u^\dagger \in C^{2+l, 1+\frac{l}{2}}(Q) \). Observing that
$f^\uparrow \geq 0$ in $Q$ and $f^\uparrow > 0$ in $U$ and using the well-known strong extremum principles for parabolic equations (see e.g. Theorem 6.1.1 (ii) in [20]), we obtain the relation

$$u^\uparrow(\cdot, T) > 0 \text{ in } \Omega(\overline{\Omega}) \text{ in case I (II).} \quad (2.46)$$

As in step 3 we assume without restriction of generality that $p \in (1, \frac{3}{2})$. The components $f^\uparrow, \beta, u^\uparrow$ of the function $\hat{f}_{u^\uparrow}$ are continuous in $\overline{Q}$. Moreover, $k_t \in L^1(0, T; L^\infty(\Omega)), k(\cdot, 0) \in L^\infty(\Omega)$ and $f \in L^p(Q)$. Thus, we have $\hat{f}_{u^\uparrow} \in L^p(Q)$. Moreover, as in step 3 we can show (by embedding theorems) that $u_0 \in W_p^{2-\frac{2}{p}}(\Omega)$ and $g \in W_p^{2-\frac{1}{p}-\beta, 1-\frac{1}{p}+\frac{\beta}{2}}(S)$. Therefore, due to Theorem 2.1 (i) the problem (2.45) has a unique solution $\hat{u} \in W_p^{2,1}(Q)$. Because of $f - f^\uparrow \geq 0$, $\beta > 0$, $k_t \leq 0$, $u^\uparrow \geq 0$, $c_k - k(\cdot, 0) \geq 0$ we have $\hat{f}_{u^\uparrow} \geq 0$. Moreover, $u_0, g \geq 0$. Thus, the part (i) of the present theorem yields $\hat{u} \geq 0$.

Adding (2.44) and (2.45) and integrating by parts the term $k_t * u^\uparrow$ we see that the function $u^\uparrow + \hat{u}$ solves the problem (2.8). By uniqueness we get $u = u^\uparrow + \hat{u}$. The relations (2.46) and $\hat{u} \geq 0$ imply the assertion (ii). Theorem is completely proved

**Remark 2.1** For $k$, depending only on $t$, the assumptions (2.25) read

$$k \in W_1^1(0, T), \quad k \geq 0, \quad k' \leq 0. \quad (2.47)$$

A natural question is: which sufficient conditions should satisfy the original relaxation kernels $m$ and $\mu$ in order to guarantee (2.47)?

The simplest particular case occurs when $m = 0$. Then $\mu = k$ and the conditions for $k$ and $\mu$ coincide. In general case, let us firstly assume that $m, \mu \in W_1^1(0, T)$ and $m, \mu \geq 0$. Then $m \in C[0, T]$ and the solution $\hat{m}$ of (2.7) also belongs to $C[0, T]$ (see e.g. [15]). From (2.7) we deduce the following relations for $\hat{m}$ and its derivative:

$$\hat{m} = m * \hat{m} + m, \quad \hat{m}' = m' * \hat{m} + m(0)\hat{m} + m'. \quad (2.48)$$

Since $m' \in L^1(0, T)$ and $m \in C[0, T]$, the right-hand side of the second equality in (2.48) belongs to $L^1(0, T)$. Therefore, $\hat{m} \in W_1^1(0, T)$. Iterating the first relation in (2.48) we get the representation of $\hat{m}$ in the form of the Neumann series: $\hat{m} = \sum_{i=0}^{\infty} m(*m)^i$ that converges in $C[0, T]$. Due to $m \geq 0$, this series is nonnegative, hence $\hat{m} \geq 0$. From (2.9) in view of $\mu, \hat{m} \in W_1^1(0, T), \mu, \hat{m} \geq 0$ we obtain the desired relations $k \in W_1^1(0, T), k \geq 0$. It remains to deduce sufficient conditions for the relation $k' \leq 0$. To this end, let us perform the following computations. Observing (2.48) we substitute the last addend $\hat{m}$ in (2.9) by $m * \hat{m} + m$ to get $k = m + \mu + (m + \mu) * \hat{m}$. Then we differentiate: $k' = m' + \mu' + (m' + \mu' + m(0) + \mu(0))\hat{m}$. Substituting the term $\hat{m}$ in the last addend again by $m * \hat{m} + m$ we finally
obtain $k' = l + l \ast \hat{m}$, where $l = m' + \mu' + (m(0) + \mu(0))m$. Since $\hat{m} \geq 0$, the sufficient condition for $k' \leq 0$ is $l \leq 0$. Summing up, the sufficient conditions for (2.47) in terms of the original kernels $m$ and $\mu$ are

$$m, \mu \in W_1^1(0, T), \ m, \mu \geq 0, \ m' + \mu' \leq -(m(0) + \mu(0))m.$$  (2.49)

For instance, the widely used exponential kernels $m(t) = \sum_{i=1}^{N} \alpha_i e^{-\gamma_i t}$, $\mu(t) = \sum_{i=1}^{N} \beta_i e^{-\gamma_i t}$ satisfy the conditions (2.49) provided $\alpha_i, \beta_i \geq 0, \gamma_i \geq \frac{\alpha_i}{\alpha_i + \beta_i} \sum_{j=1}^{N} (\alpha_j + \beta_j), \ i = 1, \ldots, N$.

2.3 Results for IP1

From now on, let the coefficients $\beta, a_{ij}, a_j$ depend only on $x$ and $k$ depend only on $t$.

Due to Lemma 2.1, IP1 is in the class of pairs $(z, u)$ of functions, whose second component $u$ together with its derivatives $u_t, u_{x_i}, u_{x_i x_j}$ belongs to $L^p(0, T), p > 1$, for any $x$, equivalent to the following inverse problem:

$$\beta(u_t + k \ast u_t) = Au + zr + f_0 \text{ in } Q,$$

$$u = u_0 \text{ in } \Omega \times \{0\}, \ B_1 u = g \text{ in } S,$$

$$u = u_T \text{ in } \Omega \times \{T\},$$  (2.50)

where $B_1, g$ are given by (2.11), (2.12) and

$$r = \phi + \hat{m} \ast \phi, \ f_0 = \chi_0 - \beta \mu u_0 + \hat{m} \ast (\chi_0 - \beta \mu u_0).$$  (2.51)

We continue to study the problem (2.50), (2.51).

2.3.1 Uniqueness

Firstly, we formulate a technical result.

**Lemma 2.2** Let (1.6), (1.7) hold. Assume $\beta \in \mathcal{C}^l(\Omega)$ with some $l \in (0, 1)$, $a_{ij}, a_j \in \mathcal{C}(\Omega)$, $a \in \mathcal{C}(\overline{Q})$, $a_t \in L^p(Q), k \in L^p(0, T)$ with some $p \in (1, \min\{\frac{3}{2}, \frac{2}{2-l}\})$ and the problem (2.8) has a solution $u \in W_2^1(Q)$ such that $B_1 u$ is continuous in a neighborhood of $\Gamma \times \{0\}$. If $f_t \in L^p(Q)$, $u_0 \in W_2^p(\Omega), A(0)u_0 + f(\cdot, 0) \in W_2^{2-l}(\Omega)$ and $g_t \in W_2^2 - 1 - \frac{1}{p} - \frac{e}{2}(S)$ then $u_t \in W_2^{2,1}(Q)$.

This lemma is proved in Publication I, i.e. [27], p. 30-31.

Now we formulate a uniqueness theorem for the inverse problem (2.50), (2.51).
Theorem 2.3 Let (1.6), (1.7), (2.47) hold and \( \beta, a_{ij}, a_j \in C^l(\Omega) \), \( a \in C^{l,\frac{1}{2}}(Q) \), \( a_t \in L^p(Q) \) with some \( l \in (0,1), p \in (1,\infty) \). Moreover, let \( a_t \geq 0 \) in \( Q \), \( r \in C^{l,\frac{1}{2}}(Q), r_t \in L^p(Q) \) and

\[
r \geq 0 , \quad r_t + k \ast r_t - \theta r \geq 0 \quad \text{in} \quad Q, \tag{2.53}
\]

where

\[
\theta = \sup_{x \in \Omega} \frac{a(x,T)}{\beta(x)} . \tag{2.54}
\]

Finally, assume that

for all \( x \in \Omega \) there exists an open subset \( Q_x \) of \( Q \) such that

\[
\exists t_x \in (0,T) : (x,t_x) \in Q_x \quad \text{and} \quad r_t + k \ast r_t - \theta r > 0 \quad \text{in} \quad Q_x. \tag{2.55}
\]

If \( (z,u) \in C^t(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q) \) solves (2.50), (2.51) and \( f_0, u_0, g, u_T = 0 \) then \( z = 0, u = 0 \).

Before proving this theorem we formulate and prove an additional technical lemma.

Lemma 2.3 Under the assumptions of Theorem 2.3 the following assertion is valid:

for all \( x \in \Omega \) there exists an open subset \( \hat{Q}_x \) of \( Q \) such that

\[
(x,t_x) \in \hat{Q}_x \quad \text{and} \quad r > 0 \quad \text{in} \quad \hat{Q}_x. \tag{2.56}
\]

Proof. Denote \( q = r_t + k \ast r_t - \theta r \). Integrating by parts the term \( k \ast r_t \) we can transform this equality to the following ordinary differential equation for \( r \): \( r_t + (k(0) - \theta)r = r(x,0)k - k' \ast r + q \). The solution is

\[
r = r(x,0)e^{(\theta-k(0))t} + e^{(\theta-k(0))t} \ast [r(x,0)k - k' \ast r] + e^{(\theta-k(0))t} \ast q. \tag{2.57}
\]

Let \( x \) be an arbitrary point in \( \Omega \). We can choose an open cylinder \( \hat{Q}_x = U \times (t_1, \tilde{t}_1) \subset Q_x \) such that \( (x,t_x) \in \hat{Q}_x \). Due to (2.47), the continuity of \( r \) and the inequality \( r \geq 0 \) the first two addends in the right-hand side of (2.57) are continuous nonnegative functions. Thus, \( r(y,t) \geq e^{(\theta-k(0))t} \ast q(y,t) \) for any \( (y,t) \in Q_x \). According to the assumptions (2.53), (2.55) and (1.8) it holds \( q \geq 0 \) in \( Q \) and there exists \( \varepsilon > 0 \) such that \( q \geq \varepsilon \) in \( \hat{Q}_x \). Inside the equivalence class corresponding to \( q \), we can choose such a \( q \) that satisfies \( q(y,t) \geq 0 \) for any \( (y,t) \in Q \) and \( q(y,t) \geq \varepsilon \) for any \( (y,t) \in \hat{Q}_x \). Now for any \( (y,t) \in \hat{Q}_x \) we estimate

\[
r(y,t) \geq e^{(\theta-k(0))t} \ast q(y,t) = \int_0^t e^{(\theta-k(0))(t-\tau)} q(y,\tau) d\tau
\]

\[
\geq \int_{t_1}^t e^{(\theta-k(0))(t-\tau)} q(y,\tau) d\tau \geq \min \{ 1 ; e^{(\theta-k(0))(t-t_1)} \} \varepsilon \cdot (t-t_1).
\]
This yields (2.56). ■

**Proof of Theorem 2.3.** The proof develops further a method that was previously applied for usual parabolic inverse problems with final over-determination in [21].

Suppose contrary that \( z \neq 0 \) and define \( z^+ = \frac{|z| + z}{2}, \ z^- = \frac{|z| - z}{2} \). Note that \( z^\pm \in C^d(\Omega) \). (The operation of taking absolute value preserves the Hölder-continuity of a function.) Firstly, let us show that

\[
z^+ \neq 0 \text{ and } z^- \neq 0. \tag{2.58}
\]

Let \( z^- = 0 \). Then \( z = z^+ \geq 0 \) and \( zr \geq 0 \). Moreover, by the supposition \( z \neq 0 \) and the continuity of \( z \) there exists an open ball \( U \) in \( \Omega \) such that \( z > 0 \) in \( U \). Let us choose some \( x_1 \in U \). Then, by virtue of (2.56) it holds \( zr > 0 \) in the open set \( [U \times (0, T)] \cap \hat{Q}_{x_1} \) of \( Q \). Observing the assumptions \( f_0, u_0, g = 0 \) and applying Theorem 2.2 to the solution \( u \) of the problem (2.50) we get \( u(x, T) > 0, \ x \in \Omega \). But this contradicts to the assumption \( u_T = 0 \). Similarly, we reach the contradiction in case \( z^+ = 0 \) making use of Theorem 2.2 for \(-u\).

Further, let us formulate the following problems for \( u^\pm \):

\[
\beta(u_t^\pm + k \ast u_t^\pm) = Au^\pm + z^\pm r \text{ in } Q, \ u^\pm = 0 \text{ in } \Omega \times \{0\}, \ B_1u^\pm = 0 \text{ in } S. \tag{2.59}
\]

By virtue of the assumptions, the free term of (2.59) has the smoothness property \( z^\pm r \in C^{l, \frac{l}{2}}(Q) \). In order to apply Theorem 2.1 (ii) to the problem (2.59), it remains to show the consistency condition \( z^\pm r = 0 \) in \( \Gamma \times \{0\} \) in case I. Due to \( u \in C^{2+l,1+\frac{l}{2}}(Q) \), the equation in (2.50) can be extended to \( \Gamma \times \{0\} \). In case I this implies the relation \( g_t = Au_0 + zr + f_0 \), and by \( g, u_0, f_0 = 0 \) the equality \( zr = 0 \) in \( \Gamma \times \{0\} \). Since the null-set of \( z^\pm \) is larger than the null-set of \( z \), we get the desired relation \( z^\pm r = 0 \) in \( \Gamma \times \{0\} \). Consequently, by Theorem 2.1 (ii) problems (2.59) have the unique solutions \( u^\pm \in C^{2+l,1+\frac{l}{2}}(Q) \).

Next step is to prove the following inequalities:

\[
u^\pm \geq 0, \ u^\pm(\cdot, T) > 0 \text{ in } \Omega (\overline{\Omega}) \text{ in case I (II)}, \tag{2.60}
\]

\[
u_t^\pm + k \ast u_t^\pm - \theta u^\pm \geq 0, \ (u_t^\pm + k \ast u_t^\pm - \theta u^\pm)(\cdot, T) > 0 \text{ in } \Omega (\overline{\Omega}) \text{ in case I (II)}. \tag{2.61}
\]

By \( z^\pm \geq 0 \) and (2.53) we have \( z^\pm r \geq 0 \). Moreover, since \( z^\pm \) are continuous and non-vanishing (see (2.58)), there exist open balls \( U^\pm \) of \( \Omega \) such that \( z^\pm > 0 \) in \( U^\pm \). Let us choose some \( x^\pm \in U^\pm \). By virtue of (2.56) it holds \( z^\pm r > 0 \) in the open subsets \( [U^\pm \times (0, T)] \cap \hat{Q}_{x^\pm} \) of \( Q \). Using Theorem 2.2 for solutions of problems (2.59) we immediately obtain (2.60).

Let us prove (2.61). Assume without restriction of generality that \( p \in (1, \min\{\frac{3}{2}, \frac{2}{2-l}\}) \). Then the assumptions of Lemma 2.2 are satisfied
for the solutions $u^{\pm}$ of the problems (2.59). (In particular, the assumption $z^{\pm}r(\cdot,0) \in W^{2,p}_{\nu}(\Omega)$ follows from $z^{\pm}r(\cdot,0) \in C^{l}(\Omega)$ and the inequality $l > 2 - \frac{2}{p}$.) Thus, we obtain $u^{\pm} \in W^{2,1}_{p}(Q)$. We have immediately $u^{\pm}_{tt}, u^{\pm}_{x}, u^{\pm}_{tx,x_j} \in L^{p}(\Omega)$ and by Lemma 2.1 (i) we get $k * u^{\pm}_{tt}, k * u^{\pm}_{x}, k * u^{\pm}_{tx,x_j} \in L^{p}(\Omega)$, which implies $k * u^{\pm} \in W^{2,1}_{p}(Q)$. From (2.59) we deduce the following problems for the functions $v^{\pm} = u^{\pm} + k * u^{\pm} - \theta u^{\pm} \in W^{2,1}_{p}(Q)$:

\[
\begin{align*}
\beta(v^{\pm} + k * v^{\pm}) &= Av^{\pm} + z^{\pm}[r_t + k * r_t - \theta r] + f^{\pm}_{1} \quad \text{in } Q, \\
v^{\pm} &= \frac{1}{\beta}z^{\pm}r \quad \text{in } \Omega \times \{0\}, \quad B_{1}v^{\pm} = 0 \quad \text{in } S,
\end{align*}
\]

(2.62)

where

\[
f^{\pm}_{1}(x,t) = a_{t}(x,t)u^{\pm}(x,t) + \int_{0}^{t} k'(t - \tau)(a(x,\tau) - a(x,t))u^{\pm}(x,\tau)d\tau.
\]

By virtue of the assumptions of theorem and $z^{\pm} \geq 0$ and $u^{\pm} \geq 0$ the free term and initial condition in (2.62) are non-negative. Therefore, Theorem 2.2 (i) implies $v^{\pm} \geq 0$, i.e. the left relation in (2.61). Moreover, according to (2.55) and the definition of $U^{\pm}$ we have $z^{\pm}[r_t + k * r_t - \theta r] > 0$ in the open subset $[U^{\pm} \times (0,T)] \cap Q_{x^{\pm}}$ of $Q$. Using Theorem 2.2 (ii) deduce the right relation in (2.61).

Since $u^{+}$ is continuous in $\overline{Q}$, there exists $x^{*} \in \overline{\Omega}$ such that

\[
u^{+}(x,T) \leq u^{+}(x^{*},T) \quad \text{for any } x \in \overline{\Omega}.
\]

(2.63)

Observing the relation $u = u^{+} - u^{-}$ and the assumption $u(\cdot, T) = u_{T} = 0$, we have $u^{+}(\cdot, T) = u^{-}(\cdot, T)$, and (2.63) implies

\[
u^{-}(x,T) \leq u^{-}(x^{*},T) \quad \text{for any } x \in \overline{\Omega}.
\]

(2.64)

Let us show that the point $x^{*}$ is the stationary maximum of $u^{\pm}(\cdot, T)$, i.e.

\[
\nabla u^{\pm}(x^{*},T) = 0.
\]

(2.65)

The equality (2.65) may fail only when $x^{*} \in \Gamma$. In case I we have the boundary condition $u^{+} = 0$ in $\Gamma$, hence in view of (2.60) the function $u^{+}(\cdot, T)$ cannot achieve its maximum on $\Gamma$, and we automatically get (2.65).

It remains to show (2.65) for the case II when $x^{*} \in \Gamma$. In this case due the vanishing boundary condition we have $\omega(x^{*}) \cdot \nabla u^{+}(x^{*},T) = 0$. (Recall that $\omega(x)$ is an outer direction at $x \in \Gamma$). Furthermore, since $u^{*}(\cdot, T)$ achieves its maximum over $\Gamma$ in the point $x = x^{*}$, we have $\tau \cdot \nabla u^{+}(x^{*},T) = 0$, where $\tau$ is any tangential direction at $x^{*}$ (this applies when $n > 1$). Summing up, we get $\xi \cdot \nabla u^{+}(x^{*},T) = 0$, where $\xi$ is any direction. This yields (2.65).
Now we are ready to present the final part of the proof. By the definitions of \( z^+ \) and \( z^- \), it holds either \( z^+(x^*) = 0 \) or \( z^-(x^*) = 0 \). In case \( z^+(x^*) = 0 \) we have \((z^+ r)(x^*, T) = 0\) and from the equation (2.59) we obtain
\[
[\beta(u^+_i + k \ast u^+_i) - au^+] (x^*, T) = A_0u^+(x^*, T),
\]
where \( A_0 = A - a \). The left-hand side of (2.66) is strictly positive due to the inequalities (1.7), (2.60), (2.61) and the definition of \( \theta \). Indeed:
\[
[\beta(u^+_i + k \ast u^+_i) - au^+] (x^*, T) = \beta(x^*) \left[ u^+_i + k \ast u^+_i - \frac{a}{\beta} u^+ \right] (x^*, T) \\
\geq \beta_0 \left[ u^+_i + k \ast u^+_i - \theta u^+ \right] (x^*, T) > 0.
\]
Therefore, the right-hand side of (2.66) is also strictly positive, i.e.
\[
A_0u^+(x^*, T) > 0. \quad (2.67)
\]
On the other hand, since \( x = x^* \) is the stationary maximum point of \( u^+(\cdot, T) \) and the principal part of \( A_0 \) is elliptic (see (1.6)), we obtain
\[
A_0u^+(x^*, T) = \sum_{i,j=1}^n a_{ij}(x^*)u^+_{x_i x_j}(x^*, T) + \sum_{j=1}^n a_j(x^*)u^+_{x_j}(x^*, T) \leq 0.
\]
This contradicts (2.67). Analogously we come to a contradiction in case \( z^-(x^*) = 0 \). Hence, the assumption \( z \neq 0 \) was incorrect. We have \( z = 0 \). Finally, whereas \( f_0, u_0, g = 0 \) by assumption, problem (2.50) is homogeneous. Thus, by the uniqueness of the solution (see Theorem 2.1) it holds \( u = 0 \). Proof is complete. \( \blacksquare \)

### 2.3.2 Existence and stability

Let us impose the following additional assumptions on the function \( r \):
\[
r \geq \delta \quad \text{in} \quad \overline{\Omega} \times (T - \delta, T) \quad \text{with some} \quad \delta \in (0, \frac{T}{2}) \quad \text{and} \\
r = 0 \quad \text{in} \quad \overline{\Omega} \times (0, \delta). \quad (2.68)
\]

Below we formulate and prove an existence and stability theorem for IP1. We note that a Fredholm-type result of this theorem (i.e. the assertion (i) of Theorem 3.8) was already obtained in [46], but under different assumptions. Namely, in [46] \( \mu = 0 \) was assumed and certain positivity conditions on the original kernel \( m \) were imposed. We do not need such assumptions in the assertion (i).

**Theorem 2.4** Let (1.6), (1.7) hold, \( \beta, a_{ij}, a_j \in C^l(\Omega) \), \( a \in C^{l,\frac{l}{2}}(Q) \) and \( a_t \in L^p(Q) \) with some \( l \in (0,1), p \in (1,\infty) \). Moreover, let \( a_t \geq 0 \),
Theorem 2.3. Indeed, the assumptions of (ii) contain the assumptions of the uniqueness Theorem 2.3. This implies that the homogeneous problem (2.72) has in \( C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q) \) only the trivial solution \( q^0 = 0, v^0 = 0 \). The solution satisfies the estimate (2.73).
As a result, all assumptions of (i) are satisfied, and the desired existence, uniqueness and stability result follows.

Thus, it remains to prove the assertion (i). We use a method that is based on a proved Fredholm-type result for the usual parabolic case (when $k = 0$). By means of this result we reduce the problem under consideration to an equation of the second kind, and apply the Fredholm’s alternative.

So, let us start with the case $k = 0$. We mention that the general assumptions of the theorem preceding the statement (i) (except for the assumptions imposed on $k$) are sufficient for the following Fredholm-type result: the inverse problem in case $k = 0$, i.e. the problem

$$
\beta u^1_t = Au^1 + z^1 r + f_0 \text{ in } Q,
$$
$$
u^1 = u_0 \text{ in } \Omega \times \{0\}, \quad B_1 u^1 = g \text{ in } S, \quad u^1 = u_T \text{ in } \Omega \times \{T\}
$$

has a solution $(z^1, u^1)$ in the space $X^l = C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ and the estimate

$$
\|z^1\|_l + \|u^1\|_{2+l,1+\frac{l}{2}} \leq \Lambda_1(\beta, a_{ij}, a_j, a, r)
\times \left\{\|f_0\|_{l,\frac{l}{2}} + \|u_0\|_{2+l} + \|g\|_{2+l-\vartheta,1+\frac{l}{2}-\vartheta} + \|u_T\|_{2+l} \right\}
$$

holds with some constant $\Lambda_1$ depending on the quantities shown in brackets, provided the corresponding homogeneous problem (i.e. the problem with the data $f_0, u_0, g, u_T = 0$) has only the trivial solution $z^1 = 0, u^1 = 0$. This follows directly from Theorem 1.2 in [21]. But because of the additional assumption (2.71) of (i) Theorem 2.3 (in case $k = 0$) implies that such a homogeneous problem indeed has only the trivial solution. Therefore, we can conclude that the unique solution $(z^1, u^1)$ of (2.75) exists in the space $X^l$ and the estimate (2.76) is valid.

For further discussion we introduce an additional Banach space of pairs of functions, whose second components are zero at $t = 0$: $X^l_0 = C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$, where

$$
C^{2+l,1+\frac{l}{2}}_0(Q) = \{v \in C^{2+l,1+\frac{l}{2}}(Q) : v = 0 \text{ in } \Omega \times \{0\}\}
$$

with the norm $\|v\|_{C^{2+l,1+\frac{l}{2}}_0(Q)} = \|v\|_{2+l,1+\frac{l}{2}}$.

Let us denote $q = z - z^1$ and $v = u - u^1$. Then the inverse problem (2.50), (2.51) for $(z, u) \in X^l$ is equivalent to the following inverse problem for the pair $X = (q, v) \in X^l_0$.

$$
\beta v_t = Av + qr - \beta k \ast (u^1_t + v_t) \text{ in } Q,
$$
$$
v = 0 \text{ in } \Omega \times \{0\}, \quad B_1 v = 0 \text{ in } S, \quad v = 0 \text{ in } \Omega \times \{T\}.
$$
Let $\mathcal{P}$ stand for the operator that assigns to a given right-hand side $f_0$ the solution of the problem (2.75) with $u_0, g, u_T = 0$. In view of the statements formulated before for (2.75), it holds $\mathcal{P} \in \mathcal{L}(C^{l,\frac{1}{2}}_0(Q), \mathcal{X}^l_0)$, where the space $C^{l,\frac{1}{2}}_0(Q)$ is defined in (2.24). (The domain of $\mathcal{P}$ is $C^{l,\frac{1}{2}}_0(Q)$, because the element $f_0 \in C^{l,\frac{1}{2}}(Q)$ must satisfy the consistency condition $f_0 = 0$ in $\Gamma \times \{0\}$ in case I (cf. (2.69) in case $u_0, g, u_T = 0$).) According to (2.76) $\|\mathcal{P}\| \leq \Lambda_1(\beta, a_{ij}, a_j, a, r)$.

By virtue of Lemma 2.1 (ii) and $\beta \in C^l(\Omega)$ it holds $\beta k \ast w_t \in C^{l,\frac{1}{2}}(Q)$ for any $w \in C^{2+l,1+\frac{l}{2}}(Q)$ and $\beta k \ast w_t|_{t=0} = 0$. Thus, the operator $(q, w) \rightarrow \beta k \ast w_t$ is well-defined from $\mathcal{X}^l_0$ to $C^{l,\frac{1}{2}}(Q)$, and in turn the operator $\mathcal{T}$ defined by $\mathcal{T}(q, v) = \mathcal{P}(\beta k \ast v_t)$ is well-defined from $\mathcal{X}^l_0$ to itself. Now we see that the problem (2.77) is in $\mathcal{X}^l_0$ equivalent to the operator equation

$$X = \mathcal{T}X + \Psi,$$

(2.78)

where $\Psi = \mathcal{P}(\beta k \ast u^l_0)$.

But thanks to the assumption $k' \in L^{\frac{2}{l+1}}(0, T)$ we can even extend $\mathcal{T}$ to the space $\mathcal{L}(\mathcal{U}^{l,l'}_0, \mathcal{X}^l_0)$, where $\mathcal{U}^{l,l'}_0 = C^{l'}(\Omega) \times C^{l,\frac{1}{2}}_0(Q)$,

$$C^{l,\frac{1}{2}}_0(Q) = \{v \in C^{l,\frac{1}{2}}(Q) : v = 0 \text{ in } \Omega \times \{0\}\}$$

with the norm $\|v\|_{C^{l,\frac{1}{2}}_0(Q)} = \|v\|_{l,\frac{1}{2}},$

and $l'$ is an arbitrary number in the interval $(0, l)$. Indeed, taking into account the relation $\mathcal{T}(q, v) = \mathcal{P}(\beta k \ast v_t) = \mathcal{P}(\beta k' \ast v - \beta k(0)v)$, obtained by integration by parts, and using (2.17) we deduce

$$\|\mathcal{T}(q, v)\|_{\mathcal{X}^l_0} \leq \|\mathcal{P}\|_{l,\frac{1}{2}} \beta k' \ast v + \beta k(0)v \|_{C^{l,\frac{1}{2}}_0(Q)} \leq \Lambda_2\|v\|_{l,\frac{1}{2}} \leq \Lambda_2(\|q\|_{C^{l'}(\Omega)} + \|v\|_{C^{l,\frac{1}{2}}_0(Q)}) = \Lambda_2\|\mathcal{T}(q, v)\|_{\mathcal{U}^{l,l'}_0}$$

(2.79)

with $\Lambda_2 = \Lambda_1\|\beta\|_{l_2} \{C_0\|k'\|_{L^{\frac{2}{l+1}}(0, T)} + |k(0)|\}$. This proves $\mathcal{T} \in \mathcal{L}(\mathcal{U}^{l,l'}_0, \mathcal{X}^l_0)$.

Since $\mathcal{X}^l_0$ is compactly embedded in $\mathcal{U}^{l,l'}_0$, the operator $\mathcal{T}$ is compact in $\mathcal{U}^{l,l'}_0$. Moreover, 1 is not an eigenvalue of $\mathcal{T}$, because the equation $X^0 = \mathcal{T}X^0$ is in $\mathcal{X}^l_0$ equivalent to the problem (2.72), whose solution $X^0 = (q^0, v^0)$ is zero by the assumption. Consequently, by the Fredholm’s alternative, the equation (2.78) has a unique solution in $\mathcal{X}^l_0$. This proves the existence assertion of (i).

It remains to prove (2.73). Since 1 belongs to the resolvent set of $\mathcal{T}$, it holds $(I - \mathcal{T})^{-1} \in \mathcal{L}(\mathcal{U}^{l,l'}_0)$ and $\|(I - \mathcal{T})^{-1}\|_{\mathcal{L}(\mathcal{U}^{l,l'}_0)} = \Lambda_3(\beta, a_{ij}, a_j, a, k, r)$.
with some constant $\Lambda_3$ depending on the parameters shown in brackets. Thus, from (2.78) we immediately have

$$\|X\|_{U_0^{l,t'}} = \|(I - T)^{-1}\Psi\|_{U_0^{l,t'}} \leq \Lambda_3 \|\Psi\|_{U_0^{l,t'}}. \quad (2.80)$$

Observing that $\|T\|_{L(U_0^{l,t'}, X_0^l)} \leq \Lambda_2$ (in view of (2.79)) and $\|\cdot\|_{U_0^{l,t'}} \leq C_7 \|\cdot\|_{X_0^l}$ with some constant $C_7$ (in view of the continuous embedding of $X_0^l$ in $U_0^{l,t'}$) and using (2.80) from (2.78) again we obtain

$$\|X\|_{X_0^l} \leq \|T\|_{L(U_0^{l,t'}, X_0^l)} \|X\|_{U_0^{l,t'}} + \|\Psi\|_{X_0^l} \leq \Lambda_2 \Lambda_3 \|\Psi\|_{U_0^{l,t'}} + \|\Psi\|_{X_0^l} \leq (C_7 \Lambda_2 \Lambda_3 + 1) \|\Psi\|_{X_0^l}. \quad (2.81)$$

Here $\|\Psi\|_{X_0^l} = \|P(-\beta k \ast u_1)\|_{X_0^l} \leq \|P\| \|\beta k \ast u_1\|_{C^{l+\frac{1}{2}}_{0}(Q)} \leq \Lambda_4 \|u_1\|_{2+l,1+\frac{l}{2}}$ with $\Lambda_4 = \Lambda_1 \|iC_0\| \|k\|_{L^{\frac{2}{3}}(0,T)}$ in view of (2.17). Consequently, from (2.81) we deduce

$$\|X\|_{X_0^l} \leq \Lambda_4 (C_7 \Lambda_2 \Lambda_3 + 1) \|u_1\|_{2+l,1+\frac{l}{2}}. \quad (2.82)$$

Recall that $X = (q, v)$ with $q = z - z^1$ and $v = u - u^1$. Thus, by means of (2.82) we deduce

$$\|z\|_{t} + \|u\|_{2+l,1+\frac{l}{2}} = \|(z, u)\|_{X^l} \leq \|(q, v)\|_{X^l} + \|(z^1, u^1)\|_{X^l} \leq \{\Lambda_4(C_7 \Lambda_2 \Lambda_3 + 1) + 1\} \|(z^1, u^1)\|_{X^l}.$$

Here the term $\|(z^1, u^1)\|_{X^l}$ can be estimated by (2.76). We reach the desired estimate (2.73). ■

**Remark 2.2** The relationship between the conditions (2.53)&(2.55) and (2.71) essentially depends on $\theta$.

In case $\theta \geq 0$ provided $k \geq 0$, the conditions (2.71) imply (2.53)&(2.55). Indeed, then the relations $r \geq 0$ and $r_t - \theta r \geq 0$ in (2.71) immediately yield $r_t \geq 0$ and in turn $k \ast r_t \geq 0$. Thus, adding the nonnegative term $k \ast r_t$ to the left-hand side of the inequalities in (2.71) we immediately obtain (2.53)&(2.55). Consequently, the assumptions (2.53) and (2.55) in the formulation of the statement (ii) of Theorem 3.8 are redundant in case $\theta \geq 0$.

In case $\theta < 0$ the situation is more complicated. Then it is possible to find functions $r$ that satisfy (2.71) but not (2.53)&(2.55). To construct such a counter-example, we can make use of following ideas. Firstly, we note that the kernel $k_\alpha(t) = \alpha e^{-\alpha t}$ with sufficiently large $\alpha > 0$ approaches
the Dirac delta-distribution. Indeed, let \( t \in (0, T) \) and \( w \in L^1(0, T) \) be a function satisfying \( w' \in L^1(t-\varepsilon, t) \) with some \( \varepsilon > 0 \). Then

\[
(k_\alpha * w)(t) - w(t) = \int_0^{t-\varepsilon} \alpha e^{-\alpha(t-\tau)} w(\tau) d\tau + \int_{t-\varepsilon}^t \alpha e^{-\alpha(t-\tau)} w(\tau) d\tau - w(t)
\]

\[
= \int_0^{t-\varepsilon} \alpha e^{-\alpha(t-\tau)} w(\tau) d\tau - \int_{t-\varepsilon}^t e^{-\alpha(t-\tau)} w'(\tau) d\tau - e^{-\alpha\varepsilon} w(t-\varepsilon).
\]

Note that \( \alpha e^{-\alpha(t-\tau)} w(\tau) \to 0 \) as \( \alpha \to \infty \) for a.e. \( \tau \in (0, t-\varepsilon) \) and \( |\alpha e^{-\alpha(t-\tau)} w(\tau)| \leq \alpha e^{-\alpha \varepsilon} |w(\tau)| \leq \frac{1}{\varepsilon^\alpha} |w(\tau)| \) for any \( \alpha > 0 \) and a.e. \( \tau \in (0, t-\varepsilon) \), where \( |w(\tau)| \) is integrable on \((0, t-\varepsilon)\). Due to the dominated convergence theorem we have \( \int_0^{t-\varepsilon} \alpha e^{-\alpha(t-\tau)} w(\tau) d\tau \to 0 \) as \( \alpha \to \infty \). Similarly we get \( \int_{t-\varepsilon}^t e^{-\alpha(t-\tau)} w'(\tau) d\tau \to 0 \) as \( \alpha \to \infty \). Moreover, \( e^{-\alpha\varepsilon} w(t-\varepsilon) \to 0 \) as \( \alpha \to \infty \). Consequently, \(|(k_\alpha * w)(t) - w(t)| \to 0 \) as \( \alpha \to \infty \). Now let us define the function \( r = r(t) \) in the following manner:

\[
r(t) = 0, \ t \in [0, \delta], \quad r(t) = e^{-bt}, \ t \in [T - \delta, T],
\]

\[
r(t) = \frac{e^{-b(T-\delta)}}{T-2\delta} (t-\delta), \ t \in (\delta, T - \delta),
\]

where \( b > 0 \) is a constant. Then \( r \geq 0 \) in \((0, T)\) and \( r_t \geq 0 \) in \((0, T - \delta)\). This in view of \( \theta < 0 \) implies \( r_t - \theta r \geq 0 \) in \((0, T - \delta)\). Further, let us choose \( \beta \in (-\frac{\theta}{2}, -\theta) \). Then \( [r_t - \theta r](t) = (-b - \theta)e^{-bt} > 0 \) for any \( t \in (T - \delta, T) \). The deduced inequalities show that (2.71) is valid. On the other hand, with fixed \( t \in (T - \delta, T) \) in view of the choice of \( b \) we have \( [2r_t - \theta r](t) = (-2b - \theta)e^{-bt} < 0 \). Therefore, in case of sufficiently large \( \alpha \) it holds \( [r_t + k_\alpha * r_t - \theta r](t) = [2r_t - \theta r](t) + (k_\alpha * r_t)(t) - r_t(t) < 0 \). Consequently, (2.53) fails.

**2.4 Results for inverse coefficient problems**

In this section we deal with the nonlinear coefficient-type inverse problems IP2 and IP3 making use of previously proved results concerning the linear inverse free term problem (2.50), (2.51). For this purpose, we introduce the following notation.

Let \( \tilde{F}_{\beta,a,r} \) stand for the operator that assigns to the vector \( d = (f_0, u_0, g, u_T) \) the solution of the inverse problem (2.50), (2.51). We have shown that in case \( \beta, a, r \) and also \( a_{ij}, a_j, k \) satisfy the assumptions of Theorem 3.8 (incl. the additional assumptions of the assertion (ii) of this theorem), the operator \( \tilde{F}_{\beta,a,r} \) is well-defined from the space

\[
\mathcal{D} = \{ d : d \in C^{1,\frac{1}{2}}(Q) \times C^{2+l}(\Omega) \times C^{2+l-\vartheta,1+\frac{l}{2}-\frac{\vartheta}{2}}(S) \times C^{2+l}(\Omega), \ d \text{ satisfy the consistency conditions (2.69), (2.70)} \}
\]

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to the space $C^l_+(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ and satisfies the estimate
\[
\|\hat{F}_{\beta,a,r}(f_0, u_0, g, u_T)\|_{C^l_+(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)} \leq \Lambda(\beta, a_{ij}, a_j, a, k, r) \times \left\{ \|f_0\|_{l,2} + \|u_0\|_{2+l} + \|g\|_{2+l-\vartheta,1+\frac{l}{2}-\vartheta} + \|u_T\|_{2+l} \right\}.
\] (2.83)

2.4.1 Results for IP2

Firstly, let us study IP2. Due to Lemma 2.1, IP2 is in the class of pairs $(a, u)$ of functions, whose second component $u$ together with its derivatives $u_t, u_{x_i}, u_{x_i x_j}$ belongs to $L^p(0, T)$, $p > 1$, for any $x$, equivalent to the following inverse problem:
\[
\begin{align*}
\beta(u_t + k \ast u_t) &= A_0 u + au + f \quad \text{in } Q, \\
u &= u_0 \quad \text{in } \Omega \times \{0\}, \quad B_1 u = g \quad \text{in } S, \\
u &= u_T \quad \text{in } \Omega \times \{T\},
\end{align*}
\] (2.84)

where $f$, $B_1$ and $g$ are given by (2.10) - (2.12) and
\[
A_0 u = \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{j=1}^{n} a_j u_{x_j}.
\]

We are going to prove existence, uniqueness and stability results for the inverse problem (2.84), (2.85) in spaces of pairs $(a, u)$ whose first components $a$ belong to the following sets that depend on $l$, $\beta$ and a given number $\theta \in \mathbb{R}$:
\[
A_{\beta, \theta}^l = \{ a \in C^l(\Omega) : \sup_{x \in \Omega} \frac{a(x)}{\beta(x)} \leq \theta \}.
\]

The next theorem comprises two results for (2.84), (2.85): (i) a global uniqueness; (ii) local conditional existence and stability. The meaning of the latter one is the following: assuming the existence of the solution of (2.84), (2.85) with certain data $d$, we prove the existence of solution to (2.84), (2.85) with data $\tilde{d}$ that are sufficiently close to $d$ and estimate the difference of these solutions in terms of $\tilde{d} - d$.

**Theorem 2.5** Let (1.6), (1.7) hold, $\beta, a_{ij}, a_j \in C^l(\Omega)$ with some $l \in (0, 1)$ and $\theta \in \mathbb{R}$. Then the following assertions are valid.

(i) If $k$ satisfies (2.47) and the problem (2.84), (2.85) has the solutions $(a_1, u_1) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$, $(a_2, u_2) \in A_{\beta, \theta}^l \times C^{2+l,1+\frac{l}{2}}(Q)$, where $u = u_1$ satisfies the conditions
\[
\begin{align*}
u &\geq 0, \quad u_t + k \ast u_t - \theta u \geq 0 \quad \text{in } Q, \\
\text{for all } x \in \Omega \text{ there exists an open subset } Q_x \text{ of } Q \text{ such that } (2.86) \\
\exists t_x \in (0, T) : (x, t_x) &\in Q_x \quad \text{and} \quad u_t + k \ast u_t - \theta u > 0 \quad \text{in } Q_x,
\end{align*}
\] then $a_1 = a_2$ and $u_1 = u_2$. 

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(ii) If $k$ satisfies (2.74) and (2.84), (2.85) has a solution $(a, u) \in A^l_{\beta, \theta} \times C^{2+l, 1+\frac{l}{2}}(Q)$ such that $u$ fulfills the conditions (2.86), the conditions (2.86) with $k$ replaced by 0 and the relations

$$
\begin{align*}
  u \geq \delta \ & \text{in} \ \overline{\Omega} \times (T - \delta, T), \\
  u = 0 \ & \text{in} \ \overline{\Omega} \times (0, \delta) \ \text{with some} \ \delta \in (0, \frac{T}{2}),
\end{align*}
$$

(2.87)

then for any data vector $\tilde{f}, \tilde{u}_0, \tilde{g}, \tilde{u}_T$ such that

$$
D := \|f - f\|_{L^2} + \|\tilde{u}_0\|_{2+l} + \|\tilde{g} - g\|_{2+l - \nu, 1 + \frac{l}{2}} + \|\tilde{u}_T - u_T\|_{2+l} < \frac{1}{2\lambda^2} \ \text{with} \ \lambda = \Lambda(\beta, a, a, a, k, u),
$$

(2.88)

where $\Lambda$ is the coefficient of the estimate (2.73) (note that $u_0 = 0$ due to the second relation in (2.88)),

$$
\begin{align*}
(a) \ & \tilde{u}_0 = \tilde{g}, \ \beta \tilde{g}_t = (A_0 + a)\tilde{u}_0 + \tilde{f} \ \text{in case I}, \\
& \omega \cdot \nabla \tilde{u}_0 = \tilde{g} \ \text{in case II} \ \text{in} \ \Gamma \times \{0\} \\
(b) \ & \tilde{u}_T = \tilde{g} \ \text{in case I}, \\
& \omega \cdot \nabla \tilde{u}_T = \tilde{g} \ \text{in case II} \ \text{in} \ \Gamma \times \{T\}
\end{align*}
$$

(2.89)

and $\tilde{u}_0 = 0$ in case I in $\Gamma$, the problem (2.84), (2.85) with $f_0, u_0, g, u_T$ replaced by $\tilde{f}, \tilde{u}_0, \tilde{g}, \tilde{u}_T$ has a unique solution $(\tilde{a}, \tilde{u})$ in the ball

$$
U = \left\{(\tilde{a}, \tilde{u}) : \|\tilde{a} - a\|_l + \|\tilde{u} - u\|_{2+l, 1+\frac{l}{2}} \leq \frac{1}{\lambda} \left(1 - \sqrt{1 - 2\lambda^2 D}\right)\right\}.
$$

(2.91)

**Remark 2.3** Since $\frac{1}{\lambda}(1 - \sqrt{1 - 2\lambda^2 D}) \sim \lambda D$ as $D \to 0^+$, the relation (2.91) implies that the solution operator of the problem (2.84), (2.85) is locally Lipschitz-continuous in the neighborhood of $(a, u)$.

**Proof of Theorem 2.5.** Let us prove (i). Subtracting the problems (2.84), (2.85) for the pairs $(a_1, u_1)$ and $(a_2, u_2)$ we obtain the following problem for the pair of differences $z = a_1 - a_2$, $u = u_1 - u_2$:

$$
\begin{align*}
  \beta(u_t + k \ast u_t) = (A_0 + a_2)u + z u_1 \ & \text{in} \ Q, \\
  u = 0 \ & \text{in} \ \Omega \times \{0\}, \ B_1 u = 0 \ & \text{in} \ S, \\
  u = 0 \ & \text{in} \ \Omega \times \{T\}.
\end{align*}
$$

This problem satisfies the assumptions of Theorem 2.3. Indeed, we have $a_2 \in C^l(\Omega)$ and $a_{2,t} = 0$. Moreover, due to $a_2 \in A^l_{\beta, \theta}$ it holds $\theta_2 := \sup_{x \in \Omega} \frac{a_2(x)}{\beta(x)} \leq \theta$. In view of this inequality, the assumption (2.86) for $u =$
$u_1$ holds with $\theta$ replaced by $\theta_2$. This means that the assumptions (2.53) and (2.55) are satisfied for $r = u_1$. Applying Theorem 2.3 we obtain the assertion $z = a_1 - a_2 = 0, u = u_1 - u_2 = 0$.

Let us we prove (ii). The proof is based on Banach fixed-point principle.

Note that the problem for $(\bar{a}, \bar{u}) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ is equivalent to the following problem for the differences $z = \bar{a} - a \in C^l(\Omega), w = \bar{u} - u \in C^{2+l,1+\frac{l}{2}}(Q)$:

$$
\begin{align*}
\beta(w_t + k \ast w_t) &= (A_0 + a)w + zu + f_0[zw] \quad \text{in } Q, \\
w &= \bar{u}_0 \quad \text{in } \Omega \times \{0\}, \\
B_1 w &= \bar{g} - g \quad \text{in } S, \\
w &= \bar{u}_T - u_T \quad \text{in } \Omega \times \{T\}
\end{align*}
$$

with $f_0[zw] = zw + \bar{f} - f$. Since $\bar{u}_0 = 0$ in case I in $\Gamma$, any solution $S = (z, w) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ of (2.92) belongs to the following space:

$$\mathcal{S} = \{S = (z, w) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q) : w = 0 \text{ in case I in } \Gamma \times \{0\}\}.$$

We will transform (2.92) to a fixed-point equation in $\mathcal{S}$.

Note that owing to the properties of $\beta, a, u$ and also $a_{ij}, a_j, k$, the operator $\hat{F}_{\beta,a,u}$ is well-defined from $\mathcal{D}$ to $C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$.

Let us show that the data vector of (2.92), i.e. $(f_0[zw], \bar{u}_0, \bar{g} - g, \bar{u}_T - u_T)$ belongs to $\mathcal{D}$ for any $S = (z, w) \in \mathcal{S}$. Assume $S \in \mathcal{S}$. From (2.88) and $z \in C^l(\Omega), u \in C^{2+l,1+\frac{l}{2}}(Q)$ we immediately have $f_0[zw] \in C^{l,\frac{l}{2}}(Q)$, $\bar{u}_0, \bar{u}_T - u_T \in C^{2+l}(\Omega)$ and $\bar{g} - g \in C^{2+l-\beta,1+\frac{l}{2}-\frac{\beta}{2}}(S)$. Moreover, from the smoothness of all terms of the equation (2.84) at the corners $\Gamma \times \{0\}$, $\Gamma \times \{T\}$ and the initial, boundary and final conditions in (2.84),(2.85) it follows that the consistency conditions (2.69), (2.70) (with $\beta g_t = f$ instead of $\beta g_t = Au_0 + f_0$ in case I in $\Gamma \times \{0\}$) are satisfied. Subtracting these conditions from (2.89), (2.90) and observing that $zw = 0$ in case I in $\Gamma \times \{0\}$ (see the definition of $\mathcal{S}$) we obtain the following consistency conditions:

$$
\begin{align*}
\bar{u}_0 &= \bar{g} - g, \quad (2.93)
\end{align*}
$$

Consequently, the vector $(f_0[zw], \bar{u}_0, \bar{g} - g, \bar{u}_T - u_T)$ belongs to $\mathcal{D}$.

Now we see that the operator

$$\hat{F}(S) = \hat{F}_{\beta,a,u}(zw + \bar{f} - f, \bar{u}_0, \bar{g} - g, \bar{u}_T - u_T)$$

is well-defined for any $S \in \mathcal{S}$. Moreover, it holds $\hat{F}(\mathcal{S}) \subseteq \mathcal{S}$, because the second component of the element $\hat{F}(S) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ is zero at $\Gamma \times \{0\}$ in case I.
We can conclude that the problem (2.92) is in the space $\mathcal{Y}$ equivalent to the fixed-point equation

$$S = \hat{F}(S).$$

Define the norm $\|S\| = \|z\|_l + \|w\|_{2+l,1+\frac{l}{2}}$ in $\mathcal{Y}$ and show that $\hat{F}$ is a contraction in the ball

$$U_0 = \left\{ S \in \mathcal{Y} : \|S\| \leq \varrho := \frac{1}{\lambda} \left( 1 - \sqrt{1 - 2\lambda^2 D} \right) \right\}.$$  \hspace{1cm} (2.94)

Using (2.93), (2.83) and the definitions of $D$ and $\lambda$ in (2.88) we deduce

$$\|\hat{F}(S)\| \leq \lambda \left\{ \|zw\|_{l,\frac{l}{2}} + D \right\} \leq \lambda \left\{ \|z\|_l \|w\|_{2+l,1+\frac{l}{2}} + D \right\} \leq \lambda \left\{ \frac{1}{2} \|S\|^2 + D \right\}.$$

In case $S \in U_0$ it holds $\|\hat{F}(S)\| \leq \lambda \left\{ \frac{1}{2} \varrho^2 + D \right\}$. Note that $\varrho$ defined in (2.94) solves the quadratic equation $\lambda \left\{ \frac{1}{2} \varrho^2 + D \right\} = \varrho$. Therefore, we have $\|\hat{F}(S)\| \leq \varrho$. Hence, $\hat{F}(U_0) \subseteq U_0$.

Similarly, for $S^j = (z^j, w^j)$, $j = 1, 2$, in view of the relation

$$z^1 w^1 - z^2 w^2 = \frac{z^1 + z^2}{2} (w^1 - w^2) + (z^1 - z^2) \frac{w^1 + w^2}{2}$$

(2.95) we obtain

$$\|\hat{F}(S^1) - \hat{F}(S^2)\| = \|\hat{F}_\beta,\alpha,\varphi(z^1 w^1 - z^2 w^2, 0, 0, 0)\|$$

$$\leq \lambda \left\| \frac{S^1 + S^2}{2} \right\| \|S^1 - S^2\| \leq \frac{1}{2} (\|S^1\| + \|S^2\|) \|S^1 - S^2\|.$$

In case $S^1, S^2 \in U_0$ we have $\|\hat{F}(S^1) - \hat{F}(S^2)\| \leq q \|S^1 - S^2\|$ with $q = \lambda \varrho = 1 - \sqrt{1 - 2\lambda^2 D} < 1$. Therefore, by the contraction principle, the equation $S = \hat{F}(S)$ has a unique solution in the ball $U_0$. This yields (ii). \hspace{1cm} \blacksquare

**Remark 2.4** It is possible to deduce sufficient conditions for the data of the direct problem (2.84) that imply the the conditions (2.86) and (2.87) for solution $u$. The second relation in (2.87) simply follows from uniqueness of the solution of the direct problem (2.84) restricted to $\Omega \times (0, \delta)$ if we assume $u_0 = 0$, $g = 0$ in $S \times (0, \delta)$, $f = 0$ in $\Omega \times (0, \delta)$. The first relation in (2.87) follows from Theorem 2.2 under the assumption that $u_0, g, f \geq 0$, there exists an open subset $Q_f$ of $\Omega \times (0, T - \delta)$ such that $f > 0$ in $Q_f$ and $g \geq \delta$ in $S \times (T - \delta, T)$ in case I. This theorem has to be applied for problems restricted to the domains $\Omega \times (0, T_1)$, where $T_1 \in [T - \delta, T]$ to get the desired result. Finally, the inequalities for $u_t + k * u_t - \theta u$ in (2.84) can be shown under certain assumptions on the data constructing a direct problem for $v = u_t + k * u_t - \theta u$ and applying Theorem 2.2 to this problem. Such a problem can be constructed similarly to the construction of problems (2.62) in the proof of Theorem 2.3.
2.4.2 Results for IP3

Finally, we study IP3. In view of 2.1, IP3 is in the class of pairs \((\beta, u)\) of functions, whose second component \(u\) together with its derivatives \(u_t, u_{x_i}, u_{x_ix_j}\) belongs to \(L^p(0,T), p > 1\), for any \(x\), equivalent to the problem

\[
\begin{align*}
\beta(u_t + k \ast u_t) &= Au + f \quad \text{in } Q, \\
u &= u_0 \quad \text{in } \Omega \times \{0\}, \quad B_1 u = g \quad \text{in } S, \\
u &= u_T \quad \text{in } \Omega \times \{T\},
\end{align*}
\]

(2.96)

where \(f, B_1\) and \(g\) are given by (2.10) - (2.12). We assume here \(\mu = 0\).

Let us introduce the following set for the coefficients \(\beta\) that depends on \(\beta_0 > 0\):

\[
B^l_{\beta_0} = \{\beta \in C^l(\Omega) : \inf_{x \in \Omega} \beta(x) \geq \beta_0\}
\]

and define \(\theta_{\beta_0} = \max\{0; \frac{1}{\beta_0} \sup_{x \in \Omega} a(x,T)\}\). Then we have \(\sup_{x \in \Omega} a(x,T) \leq \theta_{\beta_0}\) for any \(\beta \in B^l_{\beta_0}\).

**Theorem 2.6** Let (1.6) hold, \(a_{ij}, a_j \in C^l(\Omega)\), \(a \in C^{2+l,1+\frac{l}{2}}(Q)\), \(a_t \in L^p(Q)\) with some \(l \in (0,1), p \in (1,\infty), a_t \geq 0\) and \(\beta_0 > 0\). Then the following assertions are valid.

(i) If \(k\) satisfies (2.47), the problem (2.96), (2.97) has the solutions \((\beta_1, u_1) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)\) and \((\beta_2, u_2) \in B^l_{\beta_0} \times C^{2+l,1+\frac{l}{2}}(Q)\) such that \(u = u_1\) and \(\hat{u} := u_t + k \ast u_t\) satisfy the conditions

\[
u_t \in L^p(\Omega), \quad \hat{u} \geq 0, \quad \hat{u_t} + k \ast \hat{u_t} - \theta_{\beta_0} \hat{u} \geq 0,
\]

for all \(x \in \Omega\) there exists an open subset \(Q_x\) of \(Q\) such that \(\exists t_x \in (0,T) : (x,t_x) \in Q_x\) and \(\hat{u}_t + k \ast \hat{u}_t - \theta_{\beta_0} \hat{u} > 0\) in \(Q_x\),

then \(\beta_1 = \beta_2\) and \(u_1 = u_2\).

(ii) If \(k\) satisfies (2.74), \(Au_0 + f = 0\) in case I in \(\Gamma \times \{0\}\) and the problem (2.96), (2.97) has a solution \((\beta, u) \in B^l_{\beta_0} \times C^{2+l,1+\frac{l}{2}}(Q)\) such that \(u\) and \(\hat{u} = u_t + k \ast u_t\) fulfill the conditions (2.98) and the conditions (2.98) with \(k\) replaced by 0,

\[
\hat{u} \geq \delta \quad \text{in } \Omega \times (T - \delta, T) \quad \text{and} \\
u_t = 0 \quad \text{in } \Omega \times (0, \delta) \quad \text{with some } \delta \in (0, \frac{T}{2}),
\]

(2.99)

then for any data vector \(\tilde{f}, \tilde{u}_0, \tilde{g}, \tilde{u}_T\) such that

\[
D := \|f - \tilde{f}\|_{\infty} + \|\tilde{u}_0 - u_0\|_{2+l} + \|\tilde{g} - g\|_{2+l - \frac{1}{2} - \frac{l}{2}} + \|\tilde{u}_T - u_T\|_{2+l} < \frac{1}{2\Lambda^2(1 + \|k\|)} \quad \text{with } \Lambda(\beta, a_{ij}, a_j, a, k, \hat{u}),
\]

(2.100)
where $\Lambda$ is the coefficient of the estimate (2.73), $\|k\| = \|k\|_{C[0,T]}$ and

\begin{align}
(a) \quad \tilde{u}_0 = \tilde{g}, \quad \beta \tilde{g}_t = A\tilde{u}_0 + \tilde{f} = 0 \quad \text{in case I,} & \\
\omega \cdot \nabla \tilde{u}_0 = \tilde{g} \quad \text{in case II} \quad \text{in } \Gamma \times \{0\} \tag{2.101}
\end{align}

\begin{align}
(b) \quad \tilde{u}_T = \tilde{g} \quad \text{in case I,} & \\
\omega \cdot \nabla \tilde{u}_T = \tilde{g} \quad \text{in case II} \quad \text{in } \Gamma \times \{T\}, \tag{2.102}
\end{align}

then the problem (2.96), (2.97) with $f_0, u_0, g, w_T$ replaced by $\tilde{f}_0, \tilde{u}_0, \tilde{g}, \tilde{u}_T$ has a unique solution $(\tilde{\beta}, \tilde{u})$ in the ball

\[
\tilde{U} = \left\{ (\tilde{\beta}, \tilde{u}) : \| \tilde{\beta} - \beta \|_t + \| \tilde{u} - u \|_{2+l,1+\frac{l}{2}} \leq \frac{1}{\lambda(1+\|k\|)} \left( 1 - \sqrt{1-2\lambda^2(1+\|k\|)}D \right) \right\}. \tag{2.103}
\]

**Remark 2.5** The relation (2.103) implies that the solution operator of the problem (2.96), (2.97) is locally Lipschitz-continuous in the neighborhood of $(\beta, u)$.

**Proof of Theorem 2.6.** The proof is similar to the proof of Theorem 2.5. To prove (i), we subtract the problems for with the pairs $(\beta_1, u_1)$ and $(\beta_2, u_2)$. Then we obtain the problem (2.50), (2.51) for the difference $z = \beta_1 - \beta_2$, $u = u_1 - u_2$ that has the zero free term $f_0$, zero initial, boundary and final conditions and contains $\beta_2$ and $\hat{u} = u_{1,t} + k \ast u_{1,t}$ instead of $\beta$ and $r$, respectively. Applying Theorem 2.3 to this problem, we immediately obtain $z = 0$, $u = 0$.

Let us prove (ii). Note that $(\beta, u) \in \tilde{S}$, where

\[
\tilde{S} = \{ S = (z, w) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q) : w_t = 0 \text{ in case I in } \Gamma \times \{0\} \}.
\]

Indeed, $(\beta, u) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$, by assumption. Setting $t = 0$ in the equation (2.96) we get $\beta u_t(\cdot, 0) = A(0)u_0 + f(\cdot, 0)$. This due to the assumption $Au_0 + f = 0$ in case I in $\Gamma \times \{0\}$ and $\beta > 0$ implies $u_t = 0$ in case I in $\Gamma \times \{0\}$. Furthermore, any solution $(\tilde{\beta}, \tilde{u}) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ also belongs to $\tilde{S}$. This can be shown analogously, making use of the assumption $A\tilde{u}_0 + \tilde{f} = 0$ in case I in $\Gamma \times \{0\}$ (see (2.101)).

Subtracting the problems for $(\beta, \tilde{u})$ and $(\beta, u)$ we see that the problem for $(\tilde{\beta}, \tilde{u}) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ is equivalent to the following problem for the pair of differences $(z, w) = (\beta - \tilde{\beta}, \tilde{u} - u)$ in the space $\tilde{S}$:

\[
(\beta w_t + k \ast w_t) = Aw + z\hat{u} + \tilde{f}_0[z, w] \quad \text{in } Q, \\
w = \tilde{u}_0 - u_0 \quad \text{in } \Omega \times \{0\}, \quad B_1 w = \tilde{g} - g \quad \text{in } S, \tag{2.104} \\
w = \tilde{u}_T - u_T \quad \text{in } \Omega \times \{T\},
\]
where \( \tilde{f}_0[z, w] = z(w_t + k \ast w_t) + \tilde{f} - f \) and \( \hat{u} = u_t + k \ast u_t \), as defined above. Let us transform (2.104) to a fixed-point equation.

Note that the assumptions \( r \in C_{1+1}^{1, \frac{1}{2}}(Q), r_t \in L^p(\Omega), (2.53), (2.55), (2.68) \) and (2.71) are satisfied for the function \( r = \hat{u} \). Indeed, the relation \( u \in C_{2+1}^{1+1, \frac{1}{2}}(Q) \) and the assumption \( u_{tt} \in L^p(Q) \) with Lemma 2.1 imply \( \hat{u} \in C_{1+1}^{1, \frac{1}{2}}(Q), \hat{u}_t \in L^p(\Omega) \). Further, (2.53), (2.55), (2.71) and the first relation in (2.68) for \( r = \hat{u} \) automatically follow from the assumptions of the assertion (ii). The second relation in (2.68) follows from the second relation in (2.99). Observing also other assumptions of the present theorem, we see that the assumptions of Theorem 3.8 are satisfied for the set of parameters \( a_{ij}, a, k, \beta, a \) and \( r = \hat{u} \). This means that the operator \( \hat{F}_{\beta, a, \hat{u}} \) is well-defined from \( \mathcal{D} \) to \( C_{1}^{1}(\Omega) \times C_{2+1}^{1+1, \frac{1}{2}}(Q) \).

Further, for any \( S = (z, w) \in \tilde{\mathcal{D}} \) the data vector \( (f_0[z, w], \tilde{u}_0 - u_0, \tilde{g} - g, \tilde{u}_T - u_T) \) of the problem (2.104) belongs to \( \mathcal{D} \). This can be shown by means of arguments that are similar to arguments that we used in the proof of Theorem 2.5. In particular, the consistency condition \( \beta(\tilde{g} - g)t = A(\tilde{u}_0 - u_0) + \tilde{f}_0[z, w] \) in case I in \( \Gamma \times \{0\} \) follows from the relations \( \beta g_t = Au_0 + f, \beta \tilde{g}_t = A\tilde{u}_0 + f_0 \) and \( w_t = 0 \) in case I in \( \Gamma \times \{0\} \).

Now we see that the operator

\[
\tilde{F}(S) = \hat{F}_{\beta, a, \hat{u}}(f_0[z, w], \tilde{u}_0 - u_0, \tilde{g} - g, \tilde{u}_T - u_T) \tag{2.105}
\]

is well-defined for any \( S \in \tilde{\mathcal{D}} \). Moreover, it holds \( \tilde{F}(\tilde{\mathcal{D}}) \subseteq \tilde{\mathcal{D}} \), because time derivative of the second component \( w^1 = (z^1, w^1) = \tilde{F}(z, w) \in C_{1}^{1}(\Omega) \times C_{2+1}^{1+1, \frac{1}{2}}(Q) \) is zero at \( \Gamma \times \{0\} \) in case I for \( (z, w) \in \tilde{\mathcal{D}} \). The latter statement follows from the equality \( \beta w^1_t = A(\tilde{u}_0 - u_0) + z^1 \hat{u} + \tilde{f}_0[z, w] \) that is derived from the equation for \( w^1 \) at \( \Gamma \times \{0\} \) and the relations \( Au_0 + f = 0, A\tilde{u}_0 + \tilde{f} = 0, \hat{u} = 0, w_t = 0 \) in case I in \( \Gamma \times \{0\} \) and \( \beta > 0 \).

Summing up, the problem (2.104) is in \( \tilde{\mathcal{D}} \) equivalent to the following operator equation:

\[
S = \tilde{F}(S).
\]

We use the norm \( \|S\| = \|(z, w)\| = \|z\|_t + \|w\|_{2+1, \frac{1}{2}} \) in \( \tilde{\mathcal{D}} \) and define the following ball in \( \tilde{\mathcal{D}} \):

\[
\tilde{U}_0 = \left\{ S \in \tilde{\mathcal{D}} : \|S\| \leq \tilde{g} := \frac{1}{\lambda(1 + \|k\|)} \left( 1 - \sqrt{1 - 2\lambda^2(1 + \|k\|)D} \right) \right\}.
\]

By means of (2.105), (2.83) and (2.100) we obtain

\[
\|\tilde{F}(S)\| \leq \tilde{\lambda} \left\{ \|z(w_t + k \ast w_t)\|_{t, \frac{3}{2}} + D \right\} \leq \tilde{\lambda} \left( 1 + \|k\| \|z\|_t \|w_t\|_{t, \frac{3}{2}} + D \right) \\
\leq \tilde{\lambda} \left\{ \frac{1 + \|k\|}{\lambda^2} \|S\|^2 + D \right\}.
\]
In case $S \in \bar{U}_0$ it holds $\|\bar{F}(S)\| \leq \bar{\lambda} \left\{ \frac{1+\|k\|}{2} \bar{\varrho}^2 + D \right\}$. Since $\bar{\varrho}$ solves the quadratic equation $\bar{\lambda} \left\{ \frac{1+\|k\|}{2} \bar{\varrho}^2 + D \right\} = \bar{\varrho}$, we have $\|\bar{F}(S)\| \leq \bar{\varrho}$. Consequently, $\bar{F}(\bar{U}_0) \subseteq \bar{U}_0$.

Moreover, for $S^j = (z^j, w^j), j = 1, 2$, in view of the relation (2.95) we deduce

$$
\|\bar{F}(S^1) - \bar{F}(S^2)\| = \|\hat{F}_{\beta,a,\hat{u}}(\hat{z}^1(w_t^1 + k * w_t^1) - z^2(w_t^2 + k * w_t^2), 0, 0, 0)\|
\leq \bar{\lambda}\|z^1(w_t^1 + k * w_t^1) - z^2(w_t^2 + k * w_t^2)\|_{l,2}
= \bar{\lambda}\left|z^1 + z^2\right|(w_t^1 - w_t^2 + k * (w_t^1 - w_t^2)) + (z^1 - z^2)(\frac{w_t^1 + w_t^2}{2} + k * \frac{w_t^1 + w_t^2}{2})\|_{l,2}
\leq \bar{\lambda}(1 + \|k\|)\left\|\frac{S^1 + S^2}{2}\right\| \|S^1 - S^2\| \leq \bar{\lambda}(1 + \|k\|)\left\|\frac{S^1 + S^2}{2}\right\| \|S^1 - S^2\|.
$$

In case $S^j \in \bar{U}_0, j = 1, 2$, it holds $\|\bar{F}(S^1) - \bar{F}(S^2)\| \leq \bar{\eta}\|S^1 - S^2\|$ with $\bar{\eta} = \bar{\lambda}(1 + \|k\|)\bar{\varrho} = 1 - \sqrt{1 - 2\bar{\lambda}^2(1 + \|k\|)D} < 1$. By the contraction principle, the equation $S = \bar{F}(S)$ has a unique solution in the ball $\bar{U}_0$. This proves (ii).

**Remark 2.6** It is possible to establish conditions that guarantee (2.86) and (2.87) for the solution $u$ of the direct problem (2.96). To this end, initial-boundary value problems for involved functions $u_t, \hat{u}$ and $\hat{v} = \hat{u}_t + k * \hat{u}_t - \theta_{\beta,0}\hat{u}$ have to be constructed. The second condition in (2.99) follows if we assume the initial data and boundary data of the problem for $u_t$ to be zero for $t \in (0, \delta)$ and other conditions in (2.86) and (2.87) follow from Theorem 2.2 under proper assumptions of the data of the problems for $\hat{u}$ and $\hat{v}$. 

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3 NON-SMOOTH PROBLEMS

In this chapter we deal with inverse problems for Eq. (1.4) in case this equation holds in a weak sense and analyze quasi-solutions of these problems. Results are taken from Publications II and III.

Let (1.4) have the following form:

\[ u_t + (\mu \ast u)_t = Au - m \ast Au + f + \nabla \phi + \varphi_t \quad \text{in } Q, \quad (3.1) \]

where \( f, \varphi \) are regular scalar functions and \( \phi \) is a regular vector function. The functions \( \phi \) and \( \varphi \) may not have classical derivatives with respect to the space variables and the time, respectively. This means that the free term \( \chi = f + \nabla \phi + \varphi_t \) is generally a singular distribution. Since we are not planning to consider inverse problems to determine \( \beta \), we assume \( \beta = 1 \), for the sake of simplicity. Moreover, we assume that \( A \) is of the divergence type and has symmetric principal part, i.e.

\[
(Av)(x) = \sum_{i,j=1}^{n} (a_{ij}(x)v_{x_j})_{x_i} + a(x)v(x), \quad a_{ij} = a_{ji}.
\]

We are going to study problems with generally mixed boundary conditions. To this end, we split the boundary of \( \Omega \) into two parts. Namely, let \( \Gamma = \Gamma_1 \cup \Gamma_2 \) and we assume that \( \text{meas } \Gamma_1 \cap \Gamma_2 = 0 \) and for any \( j \in \{1;2\} \) either \( \Gamma_j = \emptyset \) or \( \text{meas } \Gamma_j > 0 \).

Let us return to Eq. (3.1) and complement it with the initial condition

\[ u = u_0 \quad \text{in } \Omega \times \{0\} \quad (3.2) \]

and the boundary conditions

\[
\begin{align*}
  u &= g \quad \text{in } \Gamma_1 \times (0,T), \\
  -\nu_A \cdot \nabla u + m \ast \nu_A \cdot \nabla u &= h + \nu \cdot \phi \quad \text{in } \Gamma_2 \times (0,T),
\end{align*}
\]

where the functions \( u_0, g, h \) are given and

\[
\nu_A = \left( \sum_{j=1}^{n} a_{ij}\nu_j \right)_{i=1,...,n}
\]

is the co-normal vector to \( \Gamma \). In case \( \Gamma_1 = \emptyset \) (or \( \Gamma_2 = \emptyset \)) the boundary condition (3.3) (or (3.4)) is omitted.

Summing up, (3.1) - (3.4) constitute a formal direct problem for the function \( u \).

Let us pose formal inverse problems. They use instant and integral data of the solution of (3.1) - (3.4).
IP4: Let the component $f$ of the free term be of the form

$$f(x, t) = f_0(x, t) + \sum_{j=1}^{N} \gamma_j(t) \omega_j(x)$$

(3.5)

and $\mu = 0$, $\varphi = 0$. Given $m, a_{ij}, a, u_0, f_0, \phi, g, h, \gamma_j, j = 1, \ldots, N$, and functions $u_{T_i}(x)$, $x \in \Omega$, $i = 1, \ldots, N$ with $0 < T_1 < T_2 < \ldots < T_N \leq T$, find $\omega_j, j = 1, \ldots, N$, such that the solution $u$ of (3.1) - (3.4) satisfies the following instant additional conditions:

$$u = u_{T_i} \quad \text{in} \quad \Omega \times \{T_i\}, \quad i = 1, 2, \ldots, N.$$

IP5: Let the component $f$ of the free term be of the form (3.5) and $\mu = 0$, $\varphi = 0$. Given $m, a_{ij}, a, f_0, \phi, g, h, \gamma_j, j = 1, \ldots, N$, and functions $v_i(x)$, $x \in \Omega$, $i = 1, \ldots, N+1$, find $\omega_j, j = 1, \ldots, N$, and $u_0$ such that the solution $u$ of (3.1) - (3.4) satisfies the following integral additional conditions:

$$\int_0^T \kappa_i(x, t) u(x, t) dt = v_i(x), \quad x \in \Omega, \quad i = 1, 2, \ldots, N+1,$$

(3.6)

where $\kappa_i, i = 1, \ldots, N+1$ are given weights.

IP6: Let $\text{meas} \Gamma_2 > 0$. Given $a_{ij}, u_0, f, \phi, \varphi, g, h$ and functions $u_T(x)$, $x \in \Omega$, $v_i(t), t \in (0, T)$, $i = 1, 2$, find $a, m$ and $\mu$ such that the solution of (3.1) - (3.4) satisfies the following final and integral additional conditions:

$$u = u_T \quad \text{in} \quad \Omega \times \{T\}$$

(3.7)

$$\int_{\Gamma_2} \kappa_i(x, t) u(x, t) d\Gamma = v_i(t), \quad t \in (0, T), \quad i = 1, 2,$$

(3.8)

where $\kappa_i, i = 1, 2$, are given weights and $d\Gamma$ is the surface measure on $\Gamma$.

Remark 3.1 In case $n = 1$ and $\Omega = (c, d)$ the integral $\int_{\Gamma_2} z(x) d\Gamma$ is merely the sum $\sum_{l=1}^{L} z(x_l)$, where $x_l \in \Gamma_2 \subseteq \{c; d\}$ and $L$ is the number of points in $\Gamma_2$ (i.e $L \in \{1; 2\}$). Then the conditions (3.8) read

$$\sum_{l=1}^{L} \kappa_i(x_l, t) u(x_l, t) = v_i(t), \quad t \in (0, T), \quad i = 1, 2.$$

(3.9)

Remark 3.2 The conditions $\varphi = 0$ and $\mu = 0$ in IP4 and IP5 is assumed for the sake of simplicity. The inclusion of generally non-vanishing $\varphi$ in IP6 is natural due to the method we will use. Namely, an adjoint problem corresponding to IP6 contains a singular time-derivative in a free term. We have to prove well-posedness results both for (3.1) - (3.4) and the adjoint problem. Therefore, it is natural to incorporate such a singular term in (3.1) already from the beginning.
3.1 Results concerning direct problem

3.1.1 Additional notation. Well-posedness of weak direct problem

In addition to cylinders $Q$, $S$ and $Q_t = \Omega \times (0, t)$, $t > 0$, defined in §1.1 and Lemma 2.1, we introduce $t$-dependent cylinders

$$ S_t = \Gamma \times (0, t), \quad \Gamma_{1,t} = \Gamma_1 \times (0, t), \quad \Gamma_{2,t} = \Gamma_2 \times (0, t) $$

for $t > 0$.

In the treatment of the weak direct problem we make use of the following functional spaces:

$$ U(Q_t) = C([0, t]; L^2(\Omega)) \cap L^2(0, t; W^1_2(\Omega)) $$

$$ U_0(Q_t) = \{ \eta \in U(Q_t) : \eta|_{\Gamma_{1,t}} = 0 \text{ in case } \Gamma_1 \neq \emptyset \} $$

$$ T(Q_t) = \{ \eta \in L^2(0, t; W^1_2(\Omega)) : \eta_t \in L^2(0, t; L^2(\Omega)) \} $$

$$ T_0(Q_t) = \{ \eta \in T(Q_t) : \eta|_{\Gamma_{1,t}} = 0 \text{ in case } \Gamma_1 \neq \emptyset \} $$

where $t \in (0, T]$. In case $t = T$ we write merely $U(Q)$, $U_0(Q)$, $T(Q)$ and $T_0(Q)$, because $Q_T = Q$.

We recall that the ellipticity condition (1.6) is assumed by default in this thesis. Let us collect other regularity assumptions on the data of the direct problem (3.1) - (3.4). They are

$$ a_{ij} \in L^\infty(\Omega), \quad \text{(3.10)} $$

$$ a \in L^{q_1}(\Omega), \quad \text{where } q_1 = 1 \text{ if } n = 1, \quad q_1 > \frac{n}{2} \text{ if } n \geq 2, \quad \text{(3.11)} $$

$$ \mu \in L^2(0, T), \quad \text{(3.12)} $$

$$ m \in L^1(0, T), \quad \text{(3.13)} $$

$$ u_0 \in L^2(\Omega), \quad \text{(3.14)} $$

$$ g \in T(Q), \quad h \in L^2(\Gamma_{2,T}), \quad \text{(3.15)} $$

$$ f \in L^2(0, T; L^{q_2}(\Omega)), \quad \text{where} $$

$$ q_2 = 1 \text{ if } n = 1, \quad q_2 \in (1, q_1) \text{ if } n = 2, \quad q_2 = \frac{2n}{n+2} \text{ if } n \geq 3, \quad \text{(3.16)} $$

$$ \phi = (\phi_1, \ldots, \phi_n) \in (L^2(Q))^n, \quad \text{(3.17)} $$

$$ \varphi \in U(Q) \quad \text{and in case } \Gamma_1 \neq \emptyset \quad \exists g_{\varphi} \in T(Q) : \varphi = g_{\varphi} \quad \text{in } \Gamma_{1,T}. \quad \text{(3.18)} $$

In case the additional conditions $a_{ij} \in W^1_2(\Omega)$, $\frac{\partial}{\partial x_i} \phi_i \in L^2(\Omega)$, $i = 1, \ldots, n$, $\varphi_t \in L^2(Q)$ hold and (3.1) - (3.4) has a classical solution $u \in L^2(Q)$ such that $u_t, u_{x_i}, u_{x_ix_j} \in L^2(Q)$, $i, j = 1, \ldots, n$, then multiplying (3.1) with
Note that this relation makes sense also in a more general case when \( a \in U \), \( \phi \) is a continuous embedding of \( W \).

**Lemma 3.1**  
The following assertions are valid:

(i) \( U(Q) \hookrightarrow L^2(0, T; L^{q_3}(\Omega)) \) where \( q_3 = \infty \) if \( n = 1 \), \( q_3 < \infty \) if \( n = 2 \) and \( q_3 = \frac{2n}{n-2} \) if \( n > 2 \); in the sequel we assume \( q_3 \in \left( \frac{n(q_1 q_2)}{q_1}, \infty \right) \) in case \( n = 2 \), where \( q_1 \) and \( q_2 \) are given in (3.11) and (3.16), respectively;

(ii) if \( a \) satisfies (3.11) then for any \( u \in L^2(0, T; L^{q_3}(\Omega)) \) it holds \( au \in L^2(0, T; L^{q_2}(\Omega)) \) and \( \| au \|_{L^2(0, T; L^{q_2}(\Omega))} \leq \hat{C}_0\| a \|_{L^{q_1}(\Omega)}\| u \|_{L^2(0, T; L^{q_3}(\Omega))} \), where \( \hat{C}_0 \) is a constant.

**Proof.** Since \( U(Q) \hookrightarrow L^2(0, T; W^{1}_2(\Omega)) \), the assertion (i) follows from the continuous embedding of \( W^{1}_2(\Omega) \) in \( L^{q_3}(\Omega) \). The assertion (ii) can be directly proved by means of the Hölder’s inequality. \( \blacksquare \)

Now we formulate and prove the main theorem of the subsection.

**Theorem 3.1**  
Let (3.10) - (3.18) hold. Then the problem (3.1) - (3.4) has a unique weak solution. This solution satisfies the estimate

\[
\| u \|_{U(Q)} \leq \hat{C}_1 \left[ \| u_0 \|_{L^2(\Omega)} + \| f \|_{L^2(0, T; L^{q_2}(\Omega))} + \| \phi \|_{L^2(\Omega)} \right]^n \\
+ \| \varphi \|_{U(Q)} + \theta_1 \{ \| g \|_{\mathcal{T}(Q)} + \| g\varphi \|_{\mathcal{T}(Q)} \} + \theta_2 \| h \|_{L^2(\Gamma_{2, T})},
\]

where \( \theta_j = 0 \) in case \( \Gamma_j = \emptyset \), \( \theta_j = 1 \) in case \( \Gamma_j \neq \emptyset \) and \( \hat{C}_1 \) is a constant depending on \( a_{ij}, a, \mu, m \).
Note that according to Remark 3.1, \( \|h\|_{L^2(\Gamma_{2},T)} = \left[ \int_{0}^{T} \sum_{l=1}^{L} h(x_l,t)^2 dt \right]^{1/2} \) in case \( n = 1 \).

**Proof.** Firstly, we prove the assertion of the theorem in case \( \mu = 0 \) and \( \varphi = 0 \).

The assertion of the theorem in case \( \mu = 0, \varphi = 0 \) and \( m = 0 \) is well-known from the theory of parabolic equations (see e.g. [43]). Let \( Z \) be the operator that assigns to the data vector \( d := (u_0, f, \phi, g, h) \) the weak solution of the problem (3.1) - (3.4) in case \( \mu = 0, \varphi = 0 \) and \( m = 0 \). Then it holds

\[
\|Zd\|_{U(Q)} \leq \hat{C}_2 \left[ \|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;L^{q_2}(\Omega))} + \|\phi\|_{(L^2(Q))^n} 
+ \theta_1 \|g\|_{T(Q)} + \theta_2 \|h\|_{L^2(\Gamma_{2},T)} \right],
\]

(3.21)

where \( \hat{C}_2 \) is a constant depending on \( a_{ij}, a \).

Further, let in general \( m \neq 0 \) and formulate the problem for the difference \( v = u - Zd \). Introducing the linear operator \( A \) by the formula

\[
Aw = Z \left( 0, -am \ast w, -\sum_{j=1}^{n} a_{ij}m \ast w_{x_j}, 0, 0 \right),
\]

the problem (3.1) - (3.4) for the weak solution \( u \in U(Q) \) is equivalent to the following operator equation for the quantity \( v \in U(Q) \):

\[
v = Av + AZd.
\]

(3.22)

To analyze this equation, have to estimate the operator \( A \). To this end, we need the following auxiliary inequality:

\[
\|m \ast y\|_{L^2(0,t;L^p(\Omega))} \leq \int_{0}^{t} \|m(t-\tau)\|_{L^2(0,\tau;L^p(\Omega))} d\tau, \ t \in (0,T),
\]

(3.23)

for any \( p \geq 1 \) and \( y \in L^2(0,T;L^p(\Omega)) \). This was proved in Publication II, i.e. [42], p. 4.

Let \( t \) be an arbitrary number in the interval \( (0,T) \). As in the proof of Theorem 2.1, we introduce the cutting operator \( P_t w = \begin{cases} w & \text{in } Q_t \\ 0 & \text{in } Q \setminus Q_t \end{cases} \).

Due to the causality we have \( Z(0, P_t f, P_t \phi, 0, 0)(x,t) = Z(0, f, \phi, 0, 0)(x,t) \) for any \( (x,t) \in \Omega_t \). Using these relations, the inequalities (3.21), (3.23) and...
the boundedness of $a_{ij}$, we compute:

$$
\|A w\|_{U(Q_t)} = \| \mathcal{Z} \left( 0, -am * w, -\sum_{j=1}^{n} a_{ij} m * w_{x_j}, 0, 0 \right) \|_{U(Q_t)}
$$

$$
= \| \mathcal{Z} \left( 0, -P_t[am * w], -P_t \left[ \sum_{j=1}^{n} a_{ij} m * w_{x_j} \right], 0, 0 \right) \|_{U(Q_t)}
\leq \| \mathcal{Z} \left( 0, -P_t[am * w], -P_t \left[ \sum_{j=1}^{n} a_{ij} m * w_{x_j} \right], 0, 0 \right) \|_{U(Q)}
$$

$$
\leq \hat{C}_2 \left[ \|P_t[am * w]\|_{L^2(0,t;L^2(\Omega))} + \sum_{i=1}^{n} \|P_t[a_{ij} m * w_{x_j}]\|_{L^2(\Omega)} \right]
= \hat{C}_2 \left[ \|am * w\|_{L^2(0,t;L^2(\Omega))} + \sum_{i=1}^{n} \|a_{ij} m * w_{x_j}\|_{L^2(Q_t)} \right]
\leq \hat{C}_3 \int_{0}^{t} |m(t-\tau)| \left\{ \|aw\|_{L^2(0,\tau;L^2(\Omega))} + \|w\|_{L^2(Q_\tau)} \right\} \mathrm{d}\tau
$$

(3.24)

with some constant $\hat{C}_3$ depending on $a_{ij}, a$. Due to Lemma 3.1, obtain

$$
\|aw\|_{L^2(0,\tau;L^2(\Omega))} \leq \hat{C}_4 \|a\|_{L^1(\Omega)} \|w\|_{U(Q_\tau)}
$$

with some constant $\hat{C}_4$. Using this relation in (3.24), we arrive at the following basic estimate for $A$:

$$
\|Aw\|_{U(Q_t)} \leq \hat{C}_5 \int_{0}^{t} |m(t-\tau)| \|w\|_{U(Q_\tau)} \mathrm{d}\tau, \quad t \in (0, T),
$$

(3.25)

where the constant $\hat{C}_5$ depends on $a_{ij}, a$.

Let us define the weighted norms in $U(Q)$: $\|v\|_{\sigma} = \sup_{0 < t < T} e^{-\sigma t} \|v\|_{U(Q_t)}$ where $\sigma \geq 0$. The estimate (3.25) implies the further estimate:

$$
\|Aw\|_{\sigma} \leq \hat{C}_5 \sup_{0 < t < T} e^{-\sigma t} \int_{0}^{t} |m(t-\tau)| \|w\|_{U(Q_\tau)} \mathrm{d}\tau
= \hat{C}_5 \sup_{0 < t < T} \int_{0}^{t} e^{-\sigma(t-\tau)} |m(t-\tau)| e^{-\sigma \tau} \|w\|_{U(Q_\tau)} \mathrm{d}\tau
\leq \hat{C}_5 \int_{0}^{T} e^{-\sigma s} |m(s)| \|w\|_{\sigma} \mathrm{d}s.
$$

Since $\int_{0}^{T} e^{-\sigma s} |m(s)| \mathrm{d}s \to 0$ as $\sigma \to \infty$, there exists $\sigma_0$, depending on $\hat{C}_5$ and $m$, such that $\hat{C}_5 \int_{0}^{T} e^{-\sigma_0 s} |m(s)| \mathrm{d}s \leq \frac{1}{2}$. Thus, $\|Aw\|_{\sigma_0} \leq \frac{1}{2} \|w\|_{\sigma_0}$. The operator $A$ is a contraction in $U(Q)$. This implies the existence and uniqueness assertions of the theorem in case $\mu = 0, \varphi = 0$.

To prove the estimate (3.20) in case $\mu = 0, \varphi = 0$, we deduce from (3.22) the inequality $\|v\|_{\sigma_0} \leq \|Av\|_{\sigma_0} + \|AZd\|_{\sigma_0} \leq \frac{1}{2} \|v\|_{\sigma_0} + \|Zd\|_{\sigma_0}$. This implies $\|v\|_{\sigma_0} \leq \|Zd\|_{\sigma_0}$, hence $\|u\|_{\sigma_0} = \|v + Zd\|_{\sigma_0} \leq 2 \|Zd\|_{\sigma_0}$. Using the equivalence relations $e^{-\sigma_0 T} \|\cdot\|_0 \leq \|\cdot\|_{\sigma_0} \leq \|\cdot\|_0$, where $\|\cdot\|_0 = \|\cdot\|_{U(Q)}$,
and (3.21), we reach
\[ \|u\|_{U(Q)} \leq 2\hat{C}_2 e^{\alpha_0 T} \left[ \|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;L^{q^2}(\Omega))} + \|\phi\|_{(L^2(Q))^n} + \theta_1\|g\|_{T(Q)} + \theta_2\|h\|_{L^2(\Gamma_2,T)} \right]. \]
(3.26)
This is (3.20) in case \(\mu = 0, \phi = 0\).

Secondly, we prove the theorem in the general case when \(\mu\) and \(\phi\) may not vanish. Let us introduce the resolvent kernel \(\hat{\mu}\) of \(\mu\) that is the solution of the Volterra equation of the second kind
\[ \hat{\mu} + \mu^*\hat{\mu} = \mu \quad \text{in} \quad (0,T). \]
(3.27)
Since \(\mu \in L^2(0,T)\), the equation (3.27) has a unique solution \(\hat{\mu} \in L^2(0,T)\) [15]. Note that the relations
\[ (I - \hat{\mu}^*)(I + \mu^*) = (I + \mu^*)(I - \hat{\mu}^*) = I \]
are valid, where \(I\) is the unity operator. Moreover, we define the following one-to-one connection between functions \(u \in U(Q)\) and \(\hat{u} \in U(Q)\):
\[ \hat{u} = u + \mu^*u - \phi \iff u = \hat{u} + \phi - \hat{\mu}^*(\hat{u} + \phi) \]
(3.29)
Now we note that the problem (3.1) - (3.4) for the weak solution \(u \in U(Q)\) is equivalent to the following problem for the weak solution \(\hat{u} \in U(Q)\):
\[ \hat{u}_t = A\hat{u} - \hat{m}^*A\hat{u} + \hat{f} + \nabla \cdot \hat{\phi} \quad \text{in} \quad Q, \]
\[ \hat{u} = \hat{u}_0 \quad \text{in} \quad \Omega \times \{0\}, \]
\[ \hat{u} = \hat{g} \quad \text{in} \quad \Gamma_{1,T}, \]
\[ -\nu A \cdot \nabla \hat{u} + \hat{m}^*\nu A \cdot \nabla \hat{u} = h + \nu \cdot \hat{\phi} \quad \text{in} \quad \Gamma_{2,T}, \]
(3.30)
where
\[ \hat{m} = m + \hat{\mu} - m^*\hat{\mu}, \quad \hat{f} = f + a\phi - \hat{m}^*a\phi, \]
\[ \hat{\phi}_i = \phi_i + \sum_{j=1}^{n} a_{ij}\varphi_{x_j} - \hat{m}^*\sum_{j=1}^{n} a_{ij}\varphi_{x_j}, \]
\[ \hat{g} = g + \mu^*g - g\varphi, \quad \hat{u}_0 = u_0 - \varphi(\cdot,0). \]
This can be directly verified, inserting \(u\) by the right formula of (3.29) to (3.19) and vice versa, inserting \(\hat{u}\) by the left formula of (3.29) to the weak form of the problem (3.30).

Taking the assumptions (3.10) - (3.18) and the relation \(\hat{\mu} \in L^2(0,T)\) into account and making use of Lemma 3.1 and Young’s inequality of convolutions, we obtain \(\hat{m} \in L^1(0,T), \hat{f} \in L^2(0,T;L^{q^2}(\Omega)), \hat{\phi} \in (L^2(Q))^n, \)
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\( \hat{u}_0 \in L^2(\Omega), \hat{g} \in U(Q) \). Moreover, the estimates

\[
\begin{align*}
\| \hat{f} \|_{L^2(0,T;L^2(\Omega))} & \leq \| f \|_{L^2(0,T;L^2(\Omega))} + C_6 \| \varphi \|_{U(Q)}, \\
\| \hat{\phi} \|_{(L^2(Q))^n} & \leq \| \phi \|_{(L^2(Q))^n} + C_6 \| \varphi \|_{U(Q)}, \\
\| \hat{g} \|_{\mathcal{I}(Q)} & \leq C_6 \| g \|_{\mathcal{I}(Q)} + \| g \varphi \|_{\mathcal{I}(Q)}, \\
\| \hat{u}_0 \|_{L^2(Q)} & \leq \| u_0 \|_{L^2(Q)} + C_6 \| \varphi \|_{U(Q)},
\end{align*}
\]

(3.31)

where the constant \( \hat{C}_6 \) depends on \( a_{ij}, a, \mu, m \). Thus, the first part of the proof applied to (3.30) implies that this problem has a unique weak solution \( \hat{w} \) in \( U(Q) \). By the mentioned equivalence of the problems (3.30) and (3.1) - (3.4) we conclude that the latter one has a unique solution \( u \) in \( U(Q) \).

Finally, we write the estimate (3.26) for the solution of the problem (3.30) and use the relations (3.31) for the data in the right-hand side. This yields

\[
\| \hat{u} \|_{U(Q)} \leq \text{RHS},
\]

where RHS is the right-hand side of the estimate (3.20).

Since due to (3.29) \( \| u \|_{U(Q)} \leq (1 + \| \hat{\mu} \|_{L^1(0,T)})(\| \hat{u} \|_{U(Q)} + \| \varphi \|_{U(Q)}) \), we reach (3.20).

We note the upper integration bound \( T \) in (3.19) can be released to be any number \( t \) from the interval \([0,T]\). Namely, (3.19) is equivalent to the following problem:

\[
\begin{align*}
\int_{\Omega} \left[ (u + \mu \ast u - \varphi)(x,t) \eta(x,t) - (u_0(x) - \varphi(x,0)) \eta(x,0) \right] dx \\
- \int \int_{Q_t} (u + \mu \ast u - \varphi) \eta_t \ dx \ dt \\
+ \int \int_{Q_t} \left[ \sum_{i,j=1}^n a_{ij} (u_{x_j} - m \ast u_{x_j}) \eta_{x_i} - a(u - m \ast u) \eta \right] \ dx \ dt \\
+ \int \int_{\Gamma_{2,t}} h \eta \ d\Gamma dt - \int \int_{Q_t} (f \eta - \phi \cdot \nabla \eta) \ dx \ dt = 0, \quad t \in [0,T]
\end{align*}
\]

(3.32)

for any \( \eta \in \mathcal{T}_0(Q) \). This assertion can be proved using the standard technique defining the test function as follows:

\[
\eta^\epsilon(x,t) = \begin{cases} 
\eta(x,\tau) & \text{for } \tau \in [0,t], \\
\eta(x,\tau) \left(1 - \frac{\tau-t}{\epsilon} \right) & \text{for } \tau \in (t,t+\epsilon), \\
0 & \text{for } \tau \in [t+\epsilon,T],
\end{cases}
\]

and letting the parameter \( \epsilon \) to approach 0.

### 3.1.2 Convolutional form of weak direct problem

The test function \( \eta \) in (3.19) has stronger smoothness requirements than the solution \( u \): it must possess a regular time derivative. But in some
cases we must operate with a test function that has the same regularity as \( u \), for instance in computations with adjoint problems in next sections. Therefore, we have to generalize the formulation of the weak problem in a proper manner. Since the original problem contains time convolutions, it is convenient to present such a generalization also in a convolutional form.

**Theorem 3.2** The function \( u \in \mathcal{U}(Q) \) satisfies the relation (3.19) for any \( \eta \in \mathcal{T}_0(Q) \) if and only if it satisfies the following relation

\[
\int_{\Omega} \left( u + \mu * u - \varphi \right) * \eta \, dx - \int_{\Omega} \int_{0}^{t} (u_0(x) - \varphi(x, 0)) \eta(x, \tau) d\tau \, dx \\
+ \int_{\Omega} 1 * \left[ \sum_{i,j=1}^{n} a_{ij} (u_{x_j} - m * u_{x_j}) * \eta_{x_i} - a(u - m * u) * \eta \right] dx \\
+ \int_{\Gamma_2} h * \eta \, d\Gamma - \int_{\Omega} 1 * \left( f * \eta - \sum_{i=1}^{n} \phi_i * \eta_{x_i} \right) \, dx = 0, \quad t \in [0, T],
\]

for any \( \eta \in \mathcal{U}_0(Q) \).

**Proof.** It is sufficient to prove that \( u \in \mathcal{U}(Q) \) satisfies (3.32) for any \( \eta \in \mathcal{T}_0(Q) \) if and only if it satisfies (3.33) for any \( \eta \in \mathcal{U}_0(Q) \). Suppose that \( u \in \mathcal{U}(Q) \) satisfies (3.32) and choose an arbitrary \( \eta \in \mathcal{T}_0(Q) \). Let \( t_1 \) be an arbitrary number on the interval \([0, T]\) and choose some function \( \xi_{t_1} \in \mathcal{T}_0(Q) \) such that the relation

\[
\xi_{t_1}(x, t) = \eta(x, t_1 - t) \quad \text{for} \quad t \in [0, t_1]
\]

is valid. (For instance, it is possible to define \( \xi_{t_1} \) as a periodic function with respect to \( t \), i.e. \( \xi_{t_1}(x, t) = \eta(x, t_1 - t) \) for \( t \in [0, t_1] \), \( \xi_{t_1}(x, t) = \eta(x, t - t_1) \) for \( t \in [t_1, 2t_1] \), \( \xi_{t_1}(x, t) = \eta(x, 3t_1 - t) \) for \( t \in [2t_1, 3t_1] \) and so on.) Using the relation (3.32) with \( \eta \) replaced by \( \xi_{t_1} \) and setting there \( t = t_1 \) we obtain the equality

\[
K_1(t_1) + K_2(t_1) = 0,
\]

where

\[
K_1(t_1) = \int_{\Omega} \left[ (u + \mu * u - \varphi)(x, t_1) \eta(x, 0) - (u_0(x) - \varphi(x, 0)) \eta(x, t_1) \right] dx \\
+ \int_{\Omega} \int_{0}^{t_1} (u + \mu * u - \varphi)(x, \tau) \eta_t(x, t_1 - \tau) \, d\tau \, dx,
\]

\[
K_2(t_1) = \int_{\Omega} \left[ \sum_{i,j=1}^{n} a_{ij} (u_{x_j} - m * u_{x_j}) * \eta_{x_i} - a(u - m * u) * \eta \right] dx \\
+ \int_{\Gamma_2} h * \eta \, d\Gamma - \int_{\Omega} 1 * \left( f * \eta - \sum_{i=1}^{n} \phi_i * \eta_{x_i} \right) \, dx.
\]
Note that the time derivative of $\eta$ can be removed from $K_1$ by integration. Indeed, let $t_2 \in [0, T]$. Then with the notation $\hat{u} = u + \mu * u - \varphi$ we get
\[
\int_0^{t_2} K_1(t_1)dt_1 = \int_0^{t_2} \int_\Omega \hat{u}(x, t_1)\eta(x, 0)dxdt_1 - \int_0^{t_2} \int_\Omega \hat{u}(x, t_1)\eta(x, t_1)dxdt_1 \\
+ \int_0^{t_2} \int_\Omega \int_0^{t_1} \hat{u}(x, \tau)\eta_t(x, t_1 - \tau)d\tau dxdt_1.
\]
Changing the order of the integrals over $\tau$ and $t_1$ in the last term, we easily obtain
\[
\int_0^{t_2} K_1(t_1)dt_1 = \int_\Omega \int_0^{t_2} \hat{u}(x, \tau)\eta(x, t_2 - \tau)d\tau dx - \int_0^{t_2} \int_\Omega \hat{u}(x, 0)\eta(x, t_1)dxdt_1.
\]
Integrating now the whole equality (3.35) over $t_1$ from 0 to $t_2$, observing (3.36), (3.37) and finally re-denoting $t_2$ by $t$, we reach the desired relation (3.33). Thus, we have proved that (3.33) holds for any $\eta \in T_0(Q)$. But all terms in the right-hand side of (3.33) are well-defined for $\eta \in U_0(Q)$, too. Since $T_0(Q)$ is densely embedded in $U_0(Q)$, we conclude that (3.33) holds for any $\eta \in U_0(Q)$.

It remains to show that (3.33) implies (3.32). Suppose that $u \in U(Q)$ satisfies (3.33) and choose an arbitrary $\eta \in T_0(Q)$ and $t_1 \in [0, T]$. Again, let $\xi^{t_1}$ be a function from $T_0(Q)$ such (3.34) is valid. Inserting $\xi^{t_1}$ instead of $\eta$ into (3.33), differentiating with respect to $t$ and setting $t = t_1$ we come to the relation (3.32).

### 3.2 Quasi-solutions of inverse problems. Fréchet derivatives of cost functionals

#### 3.2.1 Quasi-solutions

1. Firstly, let us consider IP4. We look for the vector of unknowns $\omega = (\omega_1, \ldots, \omega_N)$ in the space $Z_1 = (L^2(\Omega))^N$. Assume that $\mu = 0$, $\varphi = 0$, (3.10), (3.11), (3.13) - (3.15), (3.17) hold, $f_0$ satisfies (3.16) and $\gamma_j \in L^2(0, T)$, $j = 1, \ldots, N$. Then, by Theorem 3.1, the problem (3.1) - (3.4) with $f$ of the form (3.5) has a unique weak solution $u \in U(Q)$ for any $\omega \in Z_1$. We denote this $\omega$-dependent solution by $u(x, t; \omega)$. Since $U(Q) \subset C([0, T]; L^2(\Omega))$, the traces $u(\cdot, T_i; \omega)$ belong to $L^2(\Omega)$.

Let $M \subseteq Z_1$. Assume $u_{T_i} \in L^2(\Omega)$, $i = 1, \ldots, N$. The quasi-solution of IP4 in the set $M$ is an element $\omega^* \in \arg\min_{\omega \in M} J_1(\omega)$, where $J_1$ is the following cost functional:

\[
J_1(\omega) = \sum_{i=1}^N \|u(x, T_i; \omega) - u_{T_i}(x)\|^2_{L^2(\Omega)}.
\]
Similarly we define cost functionals and quasi-solutions for other inverse problems, too.

(2) In IP5 we search for vectors \( z = (\omega, u_0) \in \mathcal{Z}_2 = (L^2(\Omega))^{N+1} \). Assume that \( \mu = 0, \varphi = 0, (3.10), (3.11), (3.13), (3.15), (3.17) \) hold, \( f_0 \) satisfies (3.16) and \( \gamma_j \in L^2(0,T), j = 1, \ldots, N \). Then the problem (3.1) - (3.4) with \( f \) of the form (3.5) has a unique weak solution \( u = u(x,t;z) \in U \) for any \( z \in \mathcal{Z}_2 \). Further, let \( M \subseteq \mathcal{Z}_2 \) and assume that \( \kappa_i \in L^\infty(Q), v_i \in L^2(\Omega), i = 1, \ldots, N + 1 \). The quasi-solution of IP5 in the set \( M \) is \( z^* \in \arg \min_{z \in M} J_2(z) \), where \( J_2 \) is the cost functional

\[
J_2(z) = \sum_{i=1}^{N+1} \left\| \int_0^T \kappa_i(\cdot,t)u(\cdot,t;z)dt - v_i \right\|^2_{L^2(\Omega)}.
\]

(3) In IP6 we look for the vector \( z = (a, m, \mu) \in \mathcal{Z}_3 = L^2(\Omega) \times (L^2(0,T))^2 \). Assume that \( n \in \{1; 2; 3\} \). This guarantees that any \( a \in L^2(\Omega) \) satisfies (3.11). Moreover, assume that (3.10), (3.14) - (3.18) hold, where \( q_2 \in (1, 2) \) in (3.16) in case \( n = 2 \). Under such assumptions the problem (3.1) - (3.4) has a unique weak solution \( u = u(x,t;z) \in U(Q) \) for any \( z \in \mathcal{Z}_3 \). The trace of this solution at \( \Gamma_{2,T} \) belongs to \( L^2(\Gamma_{2,T}) \) (in case \( n = 1 \), \( u(x_l,\cdot) \in L^2(0,T), l = 1, \ldots, L \)). Let \( M \subseteq \mathcal{Z}_3 \) and assume that \( u_T \in L^2(\Omega), \kappa_i \in L^\infty(\Gamma_{2,T}), v_i \in L^2(0,T), i = 1, 2 \). The quasi-solution of IP6 in the set \( M \) is \( z^* \in \arg \min_{z \in M} J_3(z) \), where \( J_3 \) is the cost functional

\[
J_3(z) = \| u(\cdot,T;z) - u_T \|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \left\| \int_{\Gamma_2} \kappa_i(x,\cdot)u(x,\cdot;z)d\Gamma - v_i \right\|^2_{L^2(0,T)}.
\]

3.2.2 General procedure to deduce adjoint problems

Suppose that the solution \( u \) of the direct problem (3.1) - (3.4) depends on a vector of parameters \( p \) that has to be determined in an inverse problem making use of certain measurements of \( u \). Let a quasi-solution of the inverse problem be sought by a method involving the Fréchet derivative of a cost functional (i.e. some gradient-type method). Usually in practice, a solution of a proper adjoint problem (or solutions of adjoint problems) are used to represent the Fréchet derivative.

We introduce a general procedure to deduce such adjoint problems in case \( \mu = 0 \) and \( \varphi = 0 \). Assume that \( \Delta u \) is the difference of solutions of the direct problem corresponding to a difference of the vector of the parameters \( \Delta p \). More precisely, we suppose that \( \Delta u \) is a solution of the
following problem:
\[
\Delta u_t + (\mu \ast \Delta u)_t = A\Delta u - m \ast A\Delta u + f^\dagger + \nabla \cdot \phi^\dagger \quad \text{in } Q, \\
\Delta u = \Delta u_0 \quad \text{in } \Omega \times \{0\}, \\
\Delta u = 0 \quad \text{in } \Gamma_{1,T}, \\
-\nu A \cdot \nabla \Delta u + m \ast \nu A \cdot \nabla \Delta u = h^\dagger + \nu \cdot \phi^\dagger \quad \text{in } \Gamma_{2,T},
\]
with some data \(f^\dagger, \phi^\dagger, \Delta u_0, h^\dagger\) depending on \(\Delta p\). We restrict ourselves to the case when the Dirichlet boundary condition of the state \(u\) is independent on \(p\). Therefore, \(\Delta u|_{\Gamma_{1,T}} = 0\) in \((3.38)\).

In practice, the adjoint parabolic problems are usually formulated as backward problems. In our context, it is better to pose adjoint problems in the forward form. The involved memory term with \(m\) is defined via a forward convolution and from the practical viewpoint, it is preferable to have the direct and adjoint problems in a similar form (e.g., to simplify parallelization of computations).

Namely, let an adjoint state \(\psi\) be a solution of the following problem:
\[
\psi_t + (\mu \ast \psi)_t = A\psi - m \ast A\psi + f^\circ + \nabla \cdot \phi^\circ \quad \text{in } Q, \\
\psi = u^\circ \quad \text{in } \Omega \times \{0\}, \\
\psi = 0 \quad \text{in } \Gamma_{1,T}, \\
-\nu A \cdot \nabla \psi + m \ast \nu A \cdot \nabla \psi = h^\circ + \nu \cdot \phi^\circ \quad \text{in } \Gamma_{2,T},
\]
where \(f^\circ, \phi^\circ, u^\circ\) and \(h^\circ\) are some data depending on \(\Delta u\) and the cost functional under consideration.

Assume that \((3.10), (3.11), (3.13)\) hold and the quadruplets \(f^\dagger, \phi^\dagger, \Delta u_0, h^\dagger\) and \(f^\circ, \phi^\circ, u^\circ, h^\circ\) satisfy the conditions \((3.14) - (3.17)\). Then, due to Theorem 1, the problems \((3.38)\) and \((3.39)\) have unique weak solutions in the space \(U(Q)\). Actually, it hold \(\Delta u, \psi \in U_0(Q)\) because of the homogeneous boundary conditions on \(\Gamma_{1,T}\).

Let us consider the relation \((3.33)\) for \(\Delta u\) and take the test function \(\eta = \psi\). Then we obtain for any \(t \in [0, T]\)
\[
\int_\Omega (\Delta u + \mu \ast \Delta u) \ast \psi \, dx - \int_\Omega \int_0^t \Delta u_0(x) \psi(x, \tau) \, d\tau \, dx \\
+ \int_\Omega \left[ \sum_{i,j=1}^n a_{ij} (\Delta u_{x_j} - m \ast \Delta u_{x_j}) \ast \psi_{x_i} - a(\Delta u - m \ast \Delta u) \ast \psi \right] \, dx \quad (3.40) \\
+ \int_{\Gamma_2} 1 \ast h^\dagger \ast \psi \, d\Gamma - \int_\Omega 1 \ast \left( f^\dagger \ast \psi - \sum_{i=1}^n \phi_i^\dagger \ast \psi_{x_i} \right) \, dx = 0.
\]
Secondly, let us consider \((3.33)\) for \(\psi\) and take the test function \(\eta = \Delta u\).
Then we have for any $t \in [0, T]$

$$\int_{\Omega} (\psi + \mu \ast \psi) \ast \Delta u \, dx - \int_{\Omega} \int_{0}^{t} u^\circ(x) \Delta u(x, \tau) \, d\tau \, dx$$

$$+ \int_{\Omega} 1 \ast \left[ \sum_{i,j=1}^{n} a_{ij}(\psi \ast_{x} - m \ast \psi) \ast \Delta u_{x_{i}} - a(\psi - m \ast \psi) \ast \Delta u \right] \, dx$$

$$+ \int_{\Gamma_2} 1 \ast h^\circ \ast \Delta u \, d\Gamma - \int_{\Omega} 1 \ast \left( f^\circ \ast \Delta u - \sum_{i=1}^{n} \phi_i^\circ \ast \Delta u_{x_{i}} \right) \, dx = 0.$$  \hspace{1cm} (3.41)

Subtracting (3.40) from (3.41), using the commutativity of the convolution, the symmetricity relations $a_{ij} = a_{ji}$ and differentiating with respect to $t$, we arrive at the following basic equality:

$$\int_{\Omega} u^\circ(x) \Delta u(x, t) \, dx - \int_{\Gamma_2} h^\circ \ast \Delta u \, d\Gamma + \int_{\Omega} \left( f^\circ \ast \Delta u - \sum_{i=1}^{n} \phi_i^\circ \ast \Delta u_{x_{i}} \right) \, dx$$

$$= \int_{\Omega} \Delta u_0(x) \psi(x, t) \, dx - \int_{\Gamma_2} h^\dagger \ast \psi \, d\Gamma + \int_{\Omega} \left( f^\dagger \ast \psi - \sum_{i=1}^{n} \phi_i^\dagger \ast \psi_{x_{i}} \right) \, dx,$$

$$t \in [0, T].$$  \hspace{1cm} (3.42)

This relation can be used to deduce proper representations for Fréchet derivatives of cost functionals of various inverse problems. The procedure starts with a derivation of a usual expression for the Fréchet derivative, thereupon entries of the adjoint problem $f^\circ$, $\phi^\circ$, $u^\circ$ and $h^\circ$ are chosen so that the left hand side of (3.42) equals this expression. We will demonstrate this scheme in next three subsections.

We mention that the last step of the derivation of the formula (3.42), i.e. the differentiation of the difference of (3.41) and (3.40), is problematic in case the free term of the integro-differential equation (3.40) contains a singular addend $\varphi^\dagger t$, because in this case we have to differentiate a convolution $\varphi^\dagger t \ast \psi$ that may have not a regular time derivative. However, the step of differentiation is necessary, because the antiderivative of (3.42) is useless for inverse problems with instant conditions.

### 3.2.3 Derivative of $J_2$

We start by analyzing $J_2$, because this is the simplest of the functionals $J_1, J_2, J_3$.

**Theorem 3.3** Let the assumptions listed in §3.2.1 (2) be satisfied. Then the functional $J_2$ is Fréchet differentiable in $Z_2$ and $J'_2(z) \Delta z = \langle \varrho_2, \Delta z \rangle_{Z_2}$,
where the \( z \)-dependent vector \( \varrho_2 = \varrho_2(x;z) \) consists of the components

\[
\varrho_{2,j}(x;z) = \int_0^T \gamma_j(t) \psi(x,T-t;z) dt, \quad j = 1, \ldots, N,
\]

\[
\varrho_{2,N+1}(x;z) = \psi(x,T;z),
\]

\( \psi = \psi(x,t;z) \in \mathcal{U}(Q) \) is the unique \( z \)-dependent weak solution of the following (adjoint) problem:

\[
\begin{aligned}
\psi_t &= A\psi - m^* A\psi \\
&+ 2 \sum_{i=1}^{N+1} \kappa_i(x,T-t) \left[ \int_0^T \kappa_i(x,\tau) u(x,\tau;z) d\tau - v_i(x) \right] \quad \text{in } Q, \\
\psi &= 0 \quad \text{in } \Omega \times \{0\}, \\
\psi &= 0 \quad \text{in } \Gamma_{1,T}, \\
- \nu_A \cdot \nabla \psi + m^* \nu_A \cdot \nabla \psi &= 0 \quad \text{in } \Gamma_{2,T}
\end{aligned}
\]

and \( \langle \varrho_2, z \rangle_{Z_2} = \sum_{j=1}^N \langle \varrho_{2,j}, \omega_j \rangle_{L^2(\Omega)} + \langle \varrho_{2,N+1}, u_0 \rangle_{L^2(\Omega)} \) is the inner product of \( \varrho_2 \) and \( z \) in the space \( Z_2 \).

**Proof.** Let us fix some \( z = (\omega, u_0) \) and \( \Delta z = (\Delta \omega, \Delta u_0) \) in the space \( Z_2 \). One can immediately check that it holds

\[
J_2(z + \Delta z) - J_2(z) = 2 \sum_{i=1}^{N+1} \int_\Omega \int_0^T \kappa_i(x,t) \\
\times \left[ \int_0^T \kappa_i(x,\tau) u(x,\tau;z) d\tau - v_i(x) \right] \Delta u(x,t;z) dtdx
\]

\[
+ \sum_{i=1}^{N+1} \int_\Omega \left[ \int_0^T \kappa_i(x,t) \Delta u(x,t;z) dt \right]^2 dx,
\]

where \( \Delta u(x,t;z) = u(x,t;z + \Delta z) - u(x,t;z) \in \mathcal{U}_0(Q) \) is the weak solution of the following problem:

\[
\begin{aligned}
\Delta u_t &= A\Delta u - m^* A\Delta u + \sum_{j=1}^N \gamma_j \Delta \omega_j \quad \text{in } Q, \\
\Delta u &= \Delta u_0 \quad \text{in } \Omega \times \{0\}, \\
\Delta u &= 0 \quad \text{in } \Gamma_{1,T}, \\
- \nu_A \cdot \nabla \Delta u + m^* \nu_A \cdot \nabla \Delta u &= 0 \quad \text{in } \Gamma_{2,T}.
\end{aligned}
\]

Using the Cauchy inequality, the assumptions \( \kappa_i \in L^\infty(Q) \), \( \gamma_j \in L^2(0,T) \) and the estimate (3.20) for the solution of the problem (3.46) we deduce
the relation
\[
\left| \sum_{i=1}^{N+1} \int_{\Omega} \left( \int_0^T \kappa_i(x, t) \Delta u(x, t; z) \right)^2 dt \right| dx \leq \hat{C}_7 \| \Delta u \|_{H(Q)}^2
\]
\[
\leq \hat{C}_8 \left[ \left\| \sum_{j=1}^N \gamma_j \Delta \omega_j \right\|_{L^2(0, T; L^q(\Omega))}^2 + \| \Delta u_0 \|_{L^2(\Omega)}^2 \right] \leq \hat{C}_9 \| \Delta z \|_{(L^2(\Omega))^{N+1}}^2
\]
with some constants \( \hat{C}_7, \hat{C}_8, \hat{C}_9 \). Therefore \( J_2 \) is Fréchet differentiable and the first term in the right-hand side of (3.45) represents the Fréchet derivative, i.e.
\[
J_2'(z) \Delta z = 2 \sum_{i=1}^{N+1} \int_{\Omega} \int_0^T \kappa_i(x, t) \times \left[ \int_0^T \kappa_i(x, \tau) u(x, \tau; z) d\tau - v_i(x) \right] \Delta u(x, t; z) dt dx.
\] (3.48)
Comparing (3.46) with (3.38) we see that \( f^\dagger = \sum_{j=1}^N \gamma_j \Delta \omega_j \), \( \phi^\dagger = h^\dagger = 0 \). Consequently, the relation (3.42) has the form
\[
\int_{\Omega} u^\circ(x) \Delta u(x, t) dx - \int_{\Gamma_2} h^\circ \ast \Delta u d\Gamma + \int_{\Omega} \left( f^\circ \ast \Delta u - \sum_{i=1}^n \phi_i^\circ \ast \Delta u_{x_i} \right) dx
\]
\[
= \int_{\Omega} \Delta u_0(x) \psi(x, t) dx + \sum_{j=1}^N \int_{\Omega} \gamma_j \Delta \omega_j \ast \psi dx, \quad t \in [0, T].
\] (3.49)
We note that the left hand side of (3.49) coincides with (3.48) if we define \( f^\circ \) in the in the following manner:
\[
f^\circ = 2 \sum_{i=1}^{N+1} \kappa_i(x, T - t) \left[ \int_0^T \kappa_i(x, \tau) u(x, \tau; z) d\tau - v_i(x) \right],
\]
let \( u^\circ = h^\circ = \phi^\circ = 0 \) and set \( t = T \) in (3.49). The problem (3.39) with such entries takes the form of (3.44). By Theorem 3.1, the latter one has a unique solution \( \psi \in U(Q) \). Finally, the right-hand side of (3.49) at \( t = T \) equals \( \langle \rho_2, \Delta u \rangle_{Z_2} \) where the components of \( \rho_2 \) are given by (3.43). This yields the equality \( J_2'(z) \Delta z = \langle \rho_2, \Delta z \rangle_{Z_2} \). 

3.2.4 Derivative of \( J_1 \)

**Theorem 3.4** Let the assumptions listed in §3.2.1 (1) be satisfied. Then the functional \( J_1 \) is Fréchet differentiable in \( Z_1 \) and \( J_1'(\omega) \Delta \omega = \langle \rho_1, \Delta \omega \rangle_{Z_1} \),
where the $\omega$-dependent vector $\varrho_1 = \varrho_1(x; \omega)$ consists of the components

$$
\varrho_{1,j}(x; \omega) = \sum_{i=1}^{N} \int_0^{T_i} \gamma_j(t) \psi_i(x, T_i - t; \omega) dt, \ j = 1, \ldots, N, \quad (3.50)
$$

$\psi_i = \psi_i(x, t; \omega) \in U(Q), \ i = 1, \ldots, N,$ are the unique $\omega$-dependent weak solutions of the following (adjoint) problems:

$$
\begin{align*}
\psi_{i,t} &= A\psi_i - m * A\psi_i \quad \text{in } Q_{T_i}, \\
\psi_i &= 2[u(x, T_i; \omega) - u_{T_i}(x)] \quad \text{in } \Omega \times \{0\}, \\
\psi_i &= 0 \quad \text{in } \Gamma_{1,T_i}, \\
-\nu_A \cdot \nabla \psi_i + m * \nu_A \cdot \nabla \psi_i &= 0 \quad \text{in } \Gamma_{2,T_i}
\end{align*}
$$

and $\langle \varrho_1, \omega \rangle_{Z_1} = \sum_{j=1}^{N} \langle \varrho_{1,j}, \omega_j \rangle_{L^2(\Omega)}$ is the inner product of $\varrho_1$ and $\omega$ in the space $Z_1$.

**Proof.** Let us fix some $\omega, \Delta \omega \in Z_1$. It holds

$$
J_1(\omega + \Delta \omega) - J_1(\omega) = 2 \sum_{i=1}^{N} \int_{\Omega} [u(x, T_i; \omega) - u_{T_i}(x)] \Delta u(x, T_i; \omega) dx \\
+ \sum_{i=1}^{N} \int_{\Omega} \Delta u(x, T_i; \omega)^2 dx,
$$

where $\Delta u(x, t; \omega) = u(x, t; \omega + \Delta \omega) - u(x, t; \omega) \in U_0(Q)$ is the weak solution of the following problem:

$$
\begin{align*}
\Delta u_t &= A\Delta u - m * A\Delta u + \sum_{j=1}^{N} \gamma_j \Delta \omega_j \quad \text{in } Q, \\
\Delta u &= 0 \quad \text{in } \Omega \times \{0\}, \\
\Delta u &= 0 \quad \text{in } \Gamma_{1,T}, \\
-\nu_A \cdot \nabla \Delta u + m * \nu_A \cdot \nabla \Delta u &= 0 \quad \text{in } \Gamma_{2,T}.
\end{align*}
$$

Similarly to (3.47) we obtain the estimate

$$
\sum_{i=1}^{N} \int_{\Omega} \Delta u(x, T_i; \omega)^2 dx \leq \hat{C}_{10} \|\Delta u\|_{U(Q)}^2 \leq \hat{C}_{11} \|\Delta \omega\|_{L^2(\Omega)}^N
$$

with some constants $\hat{C}_{10}, \hat{C}_{11}$. This implies that $J_1$ is Fréchet differentiable and the first term in the right-hand side of (3.52) represents the Fréchet
derivative, i.e.

$$J'_1(\omega) \Delta \omega = \sum_{i=1}^{N} \sigma_i$$

with

$$\sigma_i = 2 \int \Omega [u(x, T_i; \omega) - u_{T_i}(x)] \Delta u(x, T_i; \omega) dx.$$

We are going to deduce suitable representations for the addends $\sigma_i$. For this purpose, we make use the method presented in Subsection 3.2.2, again. Comparing (3.53) with (3.38) we see that

$$f^\dagger = \sum_{j=1}^{N} \gamma_j \Delta \omega_j, \quad \phi^\dagger = \Delta u_0 = h^\dagger = 0.$$ Therefore, the relation (3.42) reads

$$\int \Omega u^\circ(x, t) \Delta u(x, t) dx - \int_{\Gamma_2} h^\circ \Delta u d\Gamma + \int \Omega \left( f^\circ \Delta u - \sum_{i=1}^{n} \phi_i^\circ \Delta u_{x_i} \right) dx = \sum_{j=1}^{N} \int \Omega \gamma_j \Delta \omega_j \psi(x) \Delta \omega_j(x) dx,$$ (3.55)

Note that the left-hand side of (3.55) equals $\sigma_i$ if we set $u_i^\circ = 2[u(x, T_i; \omega) - u_{T_i}(x)], h^\circ = f^\circ = \phi^\circ = 0$ and $t = T_i$ in (3.55). In such a case the initial condition $u_i^\circ$ of the adjoint problem (3.39) depends on the index $i$, thus the solution $\psi$ depends also on $i$. Let us denote this solution by $\psi_i$. Rewriting (3.39) for $\psi_i$ we immediately get (3.51). Due to Theorem 3.1, the problem (3.51) has a unique solution in $U(Q)$.

From (3.55) we immediately get

$$\sigma_i = \sum_{j=1}^{N} \int_0^{T_i} \int \Omega \gamma_j(t) \psi_i(x, T_i - t; \omega) \Delta \omega_j(x) dx.$$ (3.56)

From (3.54) and (3.56) we obtain $J'_1(\omega) \Delta \omega = \langle \varrho_1, \Delta \omega \rangle_{Z_1}$, where the components of $\varrho_1$ have the form (3.50).

We point out that the formulas of the components of $\varrho_1$ (3.50) contain the solutions of the problems (3.51) in cylinders $\Omega \times (0, T_i) = Q_{T_i}$ of increasing heights $T_1 < T_2 < \ldots T_N$. It turns out that we can reduce the solution of such a family of solutions to a successive solution of a certain family of $N$ problems posed on the layers $\Omega_{T_i - T_{i-1}}, i = 1, \ldots, N$. The computational cost of the latter procedure is comparable with the cost of a solution of a single problem on $\Omega \times (0, T_N)$. Let us formulate a corresponding theorem.

**Theorem 3.5** Let the assumptions listed in §3.2.1 (1) be satisfied. The components of $\varrho_1$ can also be presented in the form

$$\varrho_{1,j}(x, \omega) = \sum_{l=1}^{N} \int_{T_{l-1}}^{T_l} \gamma_j(t) \beta_l(x, T_l - t; \omega) dt, \quad j = 1, \ldots, N,$$ (3.57)
where $\beta_l \in U_0(Q_{T_l-T_{l-1}})$ are the unique $\omega$-dependent weak solutions of the following sequence of recursive problems in the domains $Q_{T_l-T_{l-1}}$:

$$
\begin{align*}
\beta_{l,t} &= A\beta_l - m * A\beta_l - af^l - \nabla \cdot \Phi^l \quad \text{in } Q_{T_l-T_{l-1}}, \\
\beta_l &= u_0^l \quad \text{in } \Omega \times \{0\}, \\
\beta_l &= 0 \quad \text{in } \Gamma_{1,T_l-T_{l-1}}, \\
-\nu_A \cdot \nabla \beta_l + m * \nu_A \cdot \nabla \beta_l &= -\nu \cdot \Phi^l \quad \text{in } \Gamma_{2,T_l-T_{l-1}},
\end{align*}
$$

where $l = N, N-1, \ldots, 2, 1$. Here

$$
u_0^l(x) = 2 \left[ u(x, T_l; \omega) - u_T(x) \right] + \Theta_l \beta_{l+1}(x, T_{l+1} - T_l; \omega)
$$

and the function $f^l$ and the vector $\Phi^l$ are defined via $\beta_N, \beta_{N-1}, \ldots, \beta_{l+1}$ as follows:

$$
f^l = \Theta_l \sum_{k=l}^{N-1} \int_0^{T_{k+1}-T_k} m(T_k-T_l+t+\tau)\beta_{k+1}(x, T_{k+1}-T_k-\tau; \omega)d\tau,
$$

$$
\Phi^l = (\Phi^l_1, \ldots, \Phi^l_n), \quad \Phi^l_i = \sum_{j=1}^n a_{ij} \frac{\partial}{\partial x_j} f^l \quad \text{and } \Theta_N = 0, \Theta_l = 1 \quad \text{for } l < N.
$$

Proof of this quite technical result is contained in Publication II, p. 9-11. The relation between $\psi_i$ an $\beta_l$ is $\beta_l(x,t;\omega) = \sum_{i=l}^{N} \psi_i(x,T_i-T_l+t;\omega)$ for $(x,t) \in Q_{T_l-T_{l-1}}$ (formula (71) in Publication II).

3.2.5 Derivative of $J_3$

We prove the Fréchet differentiability and deduce a proper representation formula for the cost functional $J_3$ of the nonlinear inverse problem IP6 in two steps.

**Theorem 3.6** Let the assumptions listed in §3.2.1 (3) be satisfied. Then the functional $J_3$ is Fréchet differentiable in $Z_3$ and

$$J'_3(z)\Delta z = 2 \int_{\Omega} [u(x, T; z) - u_T(x)] \Delta u(x, T) dx + 2 \sum_{i=1}^{2} \int_0^T \left[ \int_{\Gamma_2} \kappa_i(y, t)u(y, t; z) d\Gamma - v_i(t) \right] \int_{\Gamma_2} \kappa_i(x, t) \Delta u(x, t) d\Gamma dt,
$$

for any where $z = (a, m, \mu), \Delta z = (\Delta a, \Delta m, \Delta \mu) \in Z_3$, where $\Delta u \in U(Q)$.
is the $z$- and $\Delta z$-dependent weak solution of the following problem:

$$
\Delta u_t + (\mu \ast \Delta u)_t = A\Delta u - m \ast A\Delta u + \Delta a[u - m \ast u] - \Delta m \ast au - \nabla \cdot \left[\Delta m \ast \sum_{j=1}^n a_{ij}u_{x_j}\right] - (\Delta \mu \ast u)_t \quad \text{in } Q,
$$

$$
\Delta u = 0 \quad \text{in } \Omega \times \{0\},
$$

$$
\Delta u = 0 \quad \text{in } \Gamma_{1,T},
$$

$$
- \nu A \cdot \nabla \Delta u + m \ast \nu A \cdot \nabla u = -\nu \cdot \left[\Delta m \ast \sum_{j=1}^n a_{ij}u_{x_j}\right] \quad \text{in } \Gamma_{2,T}.
$$

**Proof.** Let us estimate the components of the free term of the integro-differential equation in (3.59). Observing the inclusion $u \in U(Q)$, Lemma 3.1 and using the Young and Cauchy inequalities we obtain

$$
\|\Delta a[u - m \ast u] - \Delta m \ast au\|_{L^2(0,T;L^2(\Omega))} \leq \hat{c}_1\|u\|_{U(Q)} \left[ (1 + \|m\|_{L^2(0,T)})\|\Delta a\|_{L^2(\Omega)} + \|a\|_{L^2(\Omega)}\|\Delta m\|_{L^2(0,T)} \right]
$$

$$
\leq \hat{c}_2(z,u)\|\Delta z\|,
$$

where $\hat{c}_1$ is a constant, $\hat{c}_2$ is a coefficient depending on $z$, $u$ and $\|\cdot\|$ denotes the norm in $Z_3$. Taking the boundedness of $a_{ij}$ into account we similarly get

$$
\|\Delta m \ast \sum_{j=1}^n a_{ij}u_{x_j}\|_{(L^2(Q))^n} \leq \hat{c}_3\|u\|_{U(Q)}\|\Delta m\|_{L^2(0,T)}
$$

with a constant $\hat{c}_3$. Further, we estimate the term $\Delta \mu \ast u$. Since $u \in C([0,T];L^2(\Omega))$ and $\Delta \mu \in L^2(0,T)$, it is easy to check that $\Delta \mu \ast u \in C([0,T];L^2(\Omega))$ and $\|\Delta \mu \ast u\|_{C([0,T];L^2(\Omega))} \leq T^{1/2}\|u\|_{C([0,T];L^2(\Omega))}\|\Delta \mu\|_{L^2(0,T)}$. Similarly, $\|\Delta \mu \ast u\|_{L^2(0,T;W^1_2(\Omega))} \leq T^{1/2}\times\|u\|_{L^2(0,T;W^1_2(\Omega))}\|\Delta \mu\|_{L^2(0,T)}$. Putting these estimates together, we have

$$
\|\Delta \mu \ast u\|_{U(Q)} \leq T^{1/2}\|u\|_{U(Q)}\|\Delta \mu\|_{L^2(0,T)}.
$$

Since $u = g$ in $\Gamma_{1,T}$, we find that

$$
\Delta \mu \ast u = \Delta \mu \ast g \quad \text{in } \Gamma_{1,T}.
$$

Using the assumption $g \in T(Q)$ and the Young and Cauchy inequalities again, we deduce

$$
\|\Delta \mu \ast g\|_{T(Q)} = \|\Delta \mu \ast g\|_{L^2(0,T;W^1_2(\Omega))} + \|(\Delta \mu \ast g)_t\|_{L^2(0,T;L^2(\Omega))}
$$

$$
= \|\Delta \mu \ast g\|_{L^2(0,T;W^1_2(\Omega))} + \|\Delta \mu \ast g_t\|_{L^2(0,T;L^2(\Omega))}
$$

$$
+ \|\Delta \mu g(\cdot,0)\|_{L^2(0,T;L^2(\Omega))} \leq \hat{c}_4\|\Delta \mu\|_{L^2(0,T)}
$$

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with a constant \( \hat{c}_4 \). The relations (3.60) - (3.64) show that the assumptions of Theorem 3.1 are satisfied for the problem (3.59). Consequently, (3.59) has a unique weak solution \( \Delta u \in \mathcal{U}(Q) \). Moreover, applying the estimate (3.20) for the solution of (3.59) we obtain
\[
\|\Delta u\|_{\mathcal{U}(Q)} \leq \hat{C}_1 \left[ \|\Delta a[u - m * u] + \Delta m * a u\|_{L^2(0,T;L^2_2(\Omega))} \right. \\
+ \left. \|\Delta m * \sum_{j=1}^n a_{ij}u_{x_j}\|_{L^2_2(Q)} \right] + \|\Delta \mu * u\|_{\mathcal{U}(Q)} + \theta_1 \|\Delta \mu * g\|_{T(Q)}
\] (3.65)
with a coefficient \( \hat{c}_5 \) depending on \( z, u \).

Next, let us denote \( \tilde{\Delta} u = u(x, t; z + \Delta z) - u(x, t; z) \) and define \( \hat{\Delta} u = \tilde{\Delta} u - \Delta u \). Then we can represent the difference of \( J_3 \) as follows:
\[
J_3(z + \Delta z) - J_3(z) = \text{RHS} + \Theta,
\] (3.66)
where RHS is the right-hand side of the equality (3.58) and
\[
\Theta = 2 \int_{\Omega} [u(x, T) - u_T(x)] \hat{\Delta} u(x, T) dx \\
+ 2 \sum_{i=1}^2 \int_0^T \left[ \int_{\Gamma_2} \kappa_i(y, t)u(y, t)d\Gamma - v_i(t) \right] \int_{\Gamma_2} \kappa_i(x, t)\hat{\Delta} u(x, t)d\Gamma dt \\
+ \int_{\Omega} \left\{ (\Delta u + \hat{\Delta} u)(x, T) \right\} dx + \sum_{i=1}^2 \int_0^T \left\{ \int_{\Gamma_2} \kappa_i(x, t)(\Delta u + \hat{\Delta} u)(x, t)d\Gamma \right\}^2 dt.
\]

The function \( \hat{\Delta} u \) satisfies the following problem:
\[
\hat{\Delta} u_t + (\mu * \hat{\Delta} u)_t = A\hat{\Delta} u - m * A\hat{\Delta} u + \bar{f} + \hat{f} \\
+ \nabla \cdot \hat{\phi} + \nabla \cdot \hat{\phi}_t + \hat{\phi}_t \quad \text{in} \ Q, \\
\hat{\Delta} u = 0 \quad \text{in} \ \Omega \times \{0\}, \\
\hat{\Delta} u = 0 \quad \text{in} \ \Gamma_{1,T}, \\
- \nu_A \cdot \nabla \hat{\Delta} u + m * \nu_A \cdot \nabla \hat{\Delta} u = \nu \cdot \hat{\phi} + \nu \cdot \hat{\phi}_t \quad \text{in} \ \Gamma_{2,T},
\] (3.67)
where
\[
\bar{f} = \Delta a\Delta u - (m + \Delta m) * \Delta a\Delta u - \Delta m * a \Delta u - \Delta m * \Delta a u, \\
\hat{f} = \Delta a\hat{\Delta} u - (m + \Delta m) * \Delta a\hat{\Delta} u - \Delta m * a \hat{\Delta} u, \\
\bar{\phi} = -\Delta m * \sum_{j=1}^n a_{ij} \Delta u_{x_j}, \quad \hat{\phi} = -\Delta m * \sum_{j=1}^n a_{ij} \hat{\Delta} u_{x_j}, \\
\bar{\phi} = -\Delta \mu * \Delta u, \quad \hat{\phi} = -\Delta \mu * \hat{\Delta} u.
\]
Similarly to (3.60) - (3.62) we deduce the following estimates:

\[ \| \hat{f} \|_{L^2(0, T; L^2(\Omega))} \leq \hat{c}_6 \left\{ (1 + \| m \|_{L^2(0, T))} + \| \Delta m \|_{L^2(0, T))} \right\} \| \Delta u \|_{L^2(\Omega)} \times \| \Delta u \|_{U(Q)} + \| \Delta m \|_{L^2(0, T))} \| a \|_{L^2(\Omega)} \right\} \right\}, \]

\[ \| \hat{f} \|_{L^2(0, T; L^2(\Omega))} \leq \hat{c}_8(\| \Delta z \|_1 + \| \Delta z \|_2) \| \hat{\Delta u} \|_{U(Q)}, \]

\[ \| \hat{\phi} \|_{L^2(Q)} \leq \hat{c}_9(\| \Delta z \|_1 + \| \Delta z \|_2) \| \hat{\Delta u} \|_{U(Q)}, \]

\[ \| \hat{\phi} \|_{L^2(Q)} \leq T^{1/2}(\| \Delta z \|_1 + \| \Delta z \|_2) \| \hat{\Delta u} \|_{U(Q)}, \]

\[ \| \hat{\phi} \|_{U(Q)} \leq T^{1/2}(\| \Delta z \|_1 + \| \Delta z \|_2) \| \hat{\Delta u} \|_{U(Q)} \]

with some coefficients \( \hat{c}_6, \ldots, \hat{c}_9 \). Moreover, since \( \Delta u = \hat{\Delta u} = 0 \) in \( \Gamma_{1,T} \), we have \( \hat{\varphi} = \hat{\varphi} = 0 \) in \( \Gamma_{1,T} \). Applying the estimate (3.20) to the solution of the problem (3.67) we get

\[ \| \hat{\Delta u} \|_{U(Q)} \]

\[ \leq \hat{c}_{10}(z, u) \left\{ [\| \Delta z \|_1 + \| \Delta z \|_2] \| \Delta u \|_{U(Q)} + \| \hat{\Delta u} \|_{U(Q)} + \| \Delta z \|_2 \right\}, \]

with a coefficient \( \hat{c}_{10} \). Provided \( \| \Delta z \|_1 \) is sufficiently small, i.e. \( \| \Delta z \|_1 + \| \Delta z \|_2 \leq \frac{1}{2\hat{c}_{10}(z, u)} \), we have

\[ \| \hat{\Delta u} \|_{U(Q)} \leq 2\hat{c}_{10}(z, u) \left\{ [\| \Delta z \|_1 + \| \Delta z \|_2] \| \Delta u \|_{U(Q)} + \| \Delta z \|_2 \right\}. \]

Due to (3.65), this yields

\[ \| \hat{\Delta u} \|_{U(Q)} \leq \hat{c}_{11}(z, u) [\| \Delta z \|_1^2 + \| \Delta z \|_2^3] \]

with a coefficient \( \hat{c}_{11} \).

In view of (3.65), (3.68), the assumption \( \kappa_j \in L^\infty(\Gamma_{2,T}) \) and a trace theorem, the RHS and the quantity \( \Theta \) satisfy the estimates

\[ |\text{RHS}| \leq \hat{c}_{12}(z, u) \| \Delta z \|_1, \quad |\Theta| \leq \hat{c}_{13}(z, u) \sum_{l=2}^{6} \| \Delta z \|_l^{l'}, \]

where \( \hat{c}_{12} \) and \( \hat{c}_{13} \) are some coefficients. Moreover, RHS is linear with respect to \( \Delta z \). This with (3.66) shows that \( J_3 \) is Fréchet differentiable in \( Z_3 \) and \( J'_3(z) \Delta z \) equals RHS.

**Theorem 3.7** Let the assumptions listed in §3.2.1 (3) be satisfied. Moreover, assume \( g = 0 \). Then the Fréchet derivative of \( J_3 \) admits the form
$J'_3(z) \Delta z = \langle \varrho_3, \Delta z \rangle_{Z_3}$, where $\varrho_3 = \tilde{\varrho}_3 + \overline{\varrho}_3$, the $z$-dependent vectors $\tilde{\varrho}_3, \overline{\varrho}_3 \in Z_3$ have the components

\begin{align}
\tilde{\varrho}_{3,1}(x; z) &= [(u - m * u) * \psi](x, T), \quad \overline{\varrho}_{3,1} = 0, \quad (3.69) \\
\tilde{\varrho}_{3,2}(t; z) &= \int_{\Omega} \left[ \sum_{i,j=1}^{n} a_{ij} \psi_{x_i} * u_{x_j} - au * \psi \right] (x, T - t) dx, \quad \overline{\varrho}_{3,2} = 0, \quad (3.70) \\
\tilde{\varrho}_{3,3}(t; z) &= \int_{\Omega} \left[ au * \psi * (\tilde{\mu} + m - m * \tilde{\mu}) - au * \psi \\
&+ \sum_{i,j=1}^{n} a_{ij} \psi_{x_i} * u_{x_j} - \sum_{i,j=1}^{n} a_{ij} \psi_{x_i} * u_{x_j} * (\tilde{\mu} + m - m * \tilde{\mu}) \right] (x, T - t) dx,
\end{align}

$$\tilde{\varrho}_{3,3}(t; z) = -2 \int_{\Omega} \{u(x, T) - u_T(x)\} [u - \tilde{\mu} * u](x, T - t) dx$$

$$- 2 \sum_{i=1}^{2} \int_{T}^{t} \left[ \int_{\Gamma_2} \kappa_i(y, \tau) u(y, \tau) d\Gamma - v_i(\tau) \right]$$

$$\times \int_{\Gamma_2} \kappa_i(x, \tau) [u - \tilde{\mu} * u](x, \tau - t) d\Gamma d\tau,$$

$\tilde{\mu}$ is the solution of (3.27), $\psi = \psi(x, t; z) \in \mathcal{U}(Q)$ is the $z$-dependent weak solution of the following (adjoint) problem:

\begin{align}
\Delta \psi_t + (\mu * \Delta \psi)_t &= A \Delta \psi - m * A \Delta \psi \quad \text{in } Q, \\
\Delta \psi &= 2[u(x, T; z) - u_T(x)] \quad \text{in } \Omega \times \{0\}, \\
\Delta \psi &= 0 \quad \text{in } \Gamma_{1,T}, \\
- \nu_A \cdot \nabla \Delta \psi + m * \nu_A \cdot \nabla \Delta \psi &= h^o \quad \text{in } \Gamma_{2,T},
\end{align}

where

$$h^o(x, t)$$

$$= -2 \sum_{i=1}^{2} \kappa_i(x, T - t) \left[ \int_{\Gamma_2} \kappa_i(y, T - t) u(y, T - t) d\Gamma - v_i(T - t) \right]$$

$$\text{and } \langle \varrho_3, z \rangle_{Z_3} = \langle \varrho_{3,1}, a \rangle_{L^2(\Omega)} + \langle \varrho_{3,2}, m \rangle_{L^2(0,T)} + \langle \varrho_{3,3}, \mu \rangle_{L^2(0,T)} \text{ is the inner product of } \varrho_3 \text{ and } z = (a, m, \mu) \text{ in } Z_3.$$

\textbf{Proof.} We are going to make use of the method presented in Subsection 3.2.2, but firstly we have to eliminate the singular term $(\Delta \mu * u)_t$ from the integro-differential equation (3.59). For this purpose, let us define a new function $\Delta w$ via is the weak solution of (3.59) $\Delta u$ by means of the formula

$$\Delta w = \Delta u + \Delta \mu * u - \tilde{\mu} * \Delta \mu * u$$

and deduce a problem for $\Delta w$. Since $u, \Delta u \in \mathcal{U}(Q)$, we have $\Delta w \in \mathcal{U}(Q)$.

Moreover, using (3.28) it is easy to
check that \( \Delta u + \mu * \Delta u + \Delta \mu * u = \Delta w + \mu * \Delta w \). Using this relation for the time derivatives in (3.59) and the equality \( \Delta u = \Delta w - \Delta \mu * u + \hat{\mu} \Delta \mu * u \) for other terms containing \( \Delta u \) in (3.59) we see that \( \Delta w \) is the weak solution of the following problem:

\[
\begin{align*}
\Delta w_t + (\mu * \Delta w)_t &= A \Delta w - m * A \Delta w + f^\dagger + \nabla \cdot \phi^\dagger \quad \text{in } Q, \\
\Delta w &= 0 \quad \text{in } \Omega \times \{0\}, \\
\Delta w &= 0 \quad \text{in } \Gamma_{1,T}, \\
- \nu_A \cdot \nabla \Delta u + m * \nu_A \cdot \nabla \Delta u &= \nu \cdot \phi^\dagger \quad \text{in } \Gamma_{2,T},
\end{align*}
\]

(3.75)

where

\[
\begin{align*}
f^\dagger &= \Delta a[u - m * u] - a \Delta m * u - a \Delta \mu * u \\
&\quad + a \Delta \mu * u * [\hat{\mu} + m - m * \hat{\mu}], \\
\phi^\dagger_i &= - \Delta m * \sum_{j=1}^{n} a_{ij} u_{x_j} - \Delta \mu * \sum_{j=1}^{n} a_{ij} u_{x_j} \\
&\quad + \Delta \mu * \sum_{j=1}^{n} a_{ij} u_{x_j} * [\hat{\mu} + m - m * \hat{\mu}].
\end{align*}
\]

Note that the integro-differential equation (3.75) doesn’t contain a singular time derivative in its free term. In addition, let us rewrite the expression of \( J'(z) \Delta z \) (3.58) in terms of \( \Delta w \). Using the formula \( \Delta u = \Delta w - (u - \hat{\mu} * u) \Delta \mu \), again, we obtain

\[
J'_3(z) \Delta z = \sigma_1 + \sigma_2 \quad \text{with}
\]

\[
\begin{align*}
\sigma_1 &= 2 \int_{\Omega} [u(x,T) - u_T(x)] \Delta w(x,T) dx \\
&\quad + 2 \sum_{i=1}^{2} \int_{0}^{T} \left[ \int_{\Gamma_2} \kappa_i(y,t) u(y,t) d\Gamma - v_i(t) \right] \int_{\Gamma_2} \kappa_i(x,t) \Delta w(x,t) d\Gamma dt, \\
\sigma_2 &= -2 \int_{\Omega} [u(x,T) - u_T(x)] [(u - \hat{\mu} * u) \Delta \mu](x,T) dx \\
&\quad - 2 \sum_{i=1}^{2} \int_{0}^{T} \left[ \int_{\Gamma_2} \kappa_i(y,t) u(y,t) d\Gamma - v_i(t) \right] \\
&\quad \times \int_{\Gamma_2} \kappa_i(x,t) [(u - \hat{\mu} * u) \Delta \mu](x,t) d\Gamma dt.
\end{align*}
\]

The term \( \sigma_2 \) has the form of the inner product \( \sigma_2 = \langle \varrho_3, \Delta z \rangle_{Z_3} \), where \( \varrho_3 = (0,0,\varrho_{3,3}) \) and \( \varrho_{3,3} \) is given by (3.72). It remains to represent \( \sigma_1 \) in a form of the inner product. Let us make use of the basic formula (3.42) with
Δu replaced by Δw. Comparing (3.75) with (3.38) we see that Δu₀ = 0 and hᵀ = 0. Thus, (3.42) reads

$$\int_{\Omega} u^\circ(x) \Delta w(x,t) dx - \int_{\Gamma_2} h^\circ \ast \Delta w d\Gamma + \int_{\Omega} \left( f^\circ \ast \Delta w - \sum_{i=1}^{n} \phi_i^\circ \ast \Delta w_{x_i} \right) dx$$

$$= \int_{\Omega} \left( f^\dagger \ast \psi - \sum_{i=1}^{n} \phi_i^\dagger \ast \psi_{x_i} \right) dx, \quad t \in [0,T].$$

The left-hand side of this expression equals σ₁ if we choose u^\circ = 2[u(x,T) - u_T(x)], define h^\circ by (3.74), take f^\circ = 0, φ^\circ = 0 and set t = T. In this case

$$\sigma_1 = \int_{\Omega} \left( f^\dagger \ast \psi - \sum_{i=1}^{n} \phi_i^\dagger \ast \psi_{x_i} \right) dx|_{t=T}. \quad \text{Substituting here the quantities} \ f^\dagger \ \text{and} \ \phi^\dagger \ \text{and rearranging the terms we reach the relation} \ \sigma_1 = \langle \bar{g}_3, \Delta z \rangle_{Z_3}, \ \text{where the components of} \ \bar{g}_3 \ \text{are given by} \ (3.69) - (3.71). \ \text{This proves the} \ \text{assertion} \ J'_1(\overline{z}) \Delta z = \langle \overline{\phi}_3, \Delta z \rangle_{Z_3}, \ \text{where} \ \overline{\phi}_3 = \bar{g}_3 + \overline{\phi}_3.

Finally, with the mentioned choice of u^\circ, h^\circ, f^\circ and φ^\circ the adjoint problem (3.39) has the form (3.73). Since u \in U(Q) and κ_i \in L^\infty(\Gamma_2,T), by trace theorems we get u^\circ \in L^2(\Omega) and h^\circ \in L^2(\Gamma_2,T). Therefore, due to Theorem 3.1, the problem (3.73) has a unique solution ψ \in U(Q).

### 3.3 Existence of quasi-solutions

**Theorem 3.8** Let the assumptions listed in §3.2.1 (1) be satisfied and M ⊂ Z₁ be compact. Then IP4 has a quasi-solution in M. Similar assertions are valid for IP5 and IP6, too.

**Proof.** Since J₁ is bounded from below, there exists m = \inf_{\omega \in M} J₁(\omega) > -\infty. Let \omega_l \in M be a minimising sequence, i.e. \lim J₁(\omega_l) = m. By the compactness, there exists a subsequence \omega_{l_j} \in M such that \lim \omega_{l_j} = \omega^* \in M. Due to the continuity of J₁, following from the Fréchet differentiability, we have \lim J₁(\omega_{l_j}) = J₁(\omega^*). Thus, J₁(\omega^*) = m and \omega^* \in \arg \min_{\omega \in M} J₁(\omega).

The element \omega^* is a quasi-solution.

**Theorem 3.9** Let the assumptions listed in §3.2.1 (1) be satisfied and M ⊂ Z₁ be bounded, closed and convex. Then IP4 has a quasi-solution in M. The set of quasi-solutions is closed and convex. Similar assertion is valid for IP5, too.

**Proof.** The existence assertion follows from Weierstrass existence theorem (see [65], Section 2.5, Thm 2D) once we have proved that J₁ is weakly sequentially lower semicontinuous in F, i.e.

$$J₁(\omega) \leq \lim inf J₁(\omega_n) \quad \text{as} \ \omega_n \rightharpoonup \omega \ \text{in} \ M. \quad (3.76)$$
On the other hand, (3.76) is a consequence of the continuity and convexity of \( J_1 \) in \( M \) \cite{65}. As mentioned before, \( J_1 \) is continuous in \( Z_1 \). Thus, it remains to show that \( J_1 \) is convex. In view of the linearity of \( u(x, t; \omega) \) with respect to \( \omega \) and the convexity of the quadratic function we obtain

\[
J_1(\lambda \hat{\omega} + (1 - \lambda) \check{\omega}) = \sum_{i=1}^{N} \int_{0}^{T} \left\{ u(x, T_i; \lambda \hat{\omega} + (1 - \lambda) \check{\omega} - u_{T_i}(x) \right\}^2 dx
\]

\[
= \sum_{i=1}^{N} \int_{0}^{T} \left\{ \lambda u(x, T_i; \hat{\omega}) + (1 - \lambda) u(x, T_i; \check{\omega}) - u_{T_i}(x) \right\}^2 dx
\]

\[
= \lambda \sum_{i=1}^{N} \int_{0}^{T} \left\{ u(x, T_i, \hat{\omega}) - u_{T_i}(x) \right\}^2 dx
\]

\[
+ (1 - \lambda) \sum_{i=1}^{N} \int_{0}^{T} \left\{ u(x, T_i, \check{\omega}) - u_{T_i}(x) \right\}^2 dx = \lambda J_1(\hat{\omega}) + (1 - \lambda) J_1(\check{\omega})
\]

for any \( \lambda \in [0, 1] \) and \( \hat{\omega}, \check{\omega} \in Z_1 \). This shows the convexity of \( J_1 \). The closedness and convexity of the set of quasi-solutions also follows from the continuity and convexity of \( J_1 \). ■

Proof of a theorem analogous to Theorem 3.9 for IP6 is a more complicated task, because in this problem \( u(x, t; z) \) is not linear with respect to \( z \) and \( J_3 \) may not be convex. We are able prove such a result in the particular case \( n = 1 \).

**Theorem 3.10** Let the assumptions listed in §3.2.1 (3) be satisfied. Assume that \( n = 1, \Omega = (c, d), \varphi = 0, g(\cdot, 0) = 0 \) and \( M \) be bounded, closed and convex. Then IP6 has a quasi-solution in \( M \).

**Proof.** Again, the assertion of the theorem follows from Weierstrass existence theorem \cite{65} provided we are able to show that \( J_3 \) is weakly sequentially lower semi-continuous in \( M \). We will prove that \( J_3 \) is even weakly sequentially continuous in \( M \).

Let us choose some sequence \( z_k = (a_k, m_k, \mu_k) \in M \) such that \( z_k \rightharpoonup z = (a, m, \mu) \in M \). Our aim is to show that \( J_3(z_k) \rightharpoonup J_3(z) \).

Firstly, we mention that the relation \( z_k \rightharpoonup z \) immediately implies \( a_k \rightharpoonup a \) in \( L^2(c, d) \) and \( m_k \rightharpoonup m, \mu_k \rightharpoonup \mu \) in \( L^2(0, T) \). The subsequent part of the proof consists of several steps.

**1. step.** Let \( \hat{\mu} \in L^2(0, T) \) be the solution of (3.27) and \( \hat{\mu}_k \in L^2(0, T) \) be the solution of the equation

\[
\hat{\mu}_k + \mu_k \ast \hat{\mu}_k = \mu_k \quad \text{in} \quad (0, T).
\]
We are going to show that $\hat{\mu}_k \rightharpoonup \hat{\mu}$ in $L^2(0, T)$.

To prove this convergence relation, we start by verifying the boundedness of the sequence $\hat{\mu}_k$ in $L^2(0, T)$. Multiplying the equation of $\hat{\mu}$ by $e^{-\sigma t}$, $\sigma > 0$, observing that $e^{-\sigma t}(\mu_k * \mu) = (e^{-\sigma t}\mu_k) * (e^{-\sigma t}\mu)$ and estimating by means of the Young and Cauchy inequalities we get

$$
\left\|e^{-\sigma t}\hat{\mu}_k\right\|_{L^2(0,T)} \leq \left\|e^{-\sigma t}\mu_k * e^{-\sigma t}\hat{\mu}_k\right\|_{L^2(0,T)} + \left\|e^{-\sigma t}\mu_k\right\|_{L^2(0,T)} \\
\leq \left\|e^{-\sigma t}\mu_k\right\|_{L^1(0,T)} \left\|e^{-\sigma t}\hat{\mu}_k\right\|_{L^2(0,T)} + \left\|e^{-\sigma t}\mu_k\right\|_{L^2(0,T)} \\
\leq \left\|e^{-\sigma t}\hat{\mu}_k\right\|_{L^2(0,T)} + \left\|e^{-\sigma t}\mu_k\right\|_{L^2(0,T)}.
$$

Since $\left\|e^{-\sigma t}\mu_k\right\|_{L^2(0,T)} \to 0$ as $\sigma \to \infty$ and the weakly converging sequence $\mu_k$ is bounded in $L^2(0, T)$, there exists $\sigma > 0$ such that $\left\|e^{-\sigma t}\mu_k\right\|_{L^2(0,T)} \leq \frac{1}{2}$. With such a $\sigma$ we obtain

$$
\left\|e^{-\sigma t}\hat{\mu}_k\right\|_{L^2(0,T)} \leq 2\left\|e^{-\sigma t}\mu_k\right\|_{L^2(0,T)} \Rightarrow \left\|\hat{\mu}_k\right\|_{L^2(0,T)} \leq 2e^{\sigma T} \sup \left\|\mu_k\right\|_{L^2(0,T)}.
$$

This shows that the sequence $\hat{\mu}_k$ is bounded in $L^2(0, T)$.

Further, the difference $\hat{\mu}_k - \hat{\mu}$ can be expressed as

$$
\hat{\mu}_k - \hat{\mu} = \mu_k - \mu - v_k * (\mu_k - \mu),
$$

where $v_k = \hat{\mu} + \hat{\mu}_k - \hat{\mu} * \hat{\mu}_k$ is a bounded sequence in $L^2(0, T)$. With an arbitrary $\zeta \in L^2(0, T)$ we have

$$
\langle \hat{\mu}_k - \hat{\mu}, \zeta \rangle_{L^2(0,T)} = -\langle \mu_k - \mu, \zeta \rangle_{L^2(0,T)} - N_k, \quad \quad (3.77)
$$

$$
N_k = \langle v_k * (\mu_k - \mu), \zeta \rangle_{L^2(0,T)} = \int_0^T v_k(\tau) \int_0^{T-\tau} (\mu_k - \mu)(s) \zeta(\tau + s) ds d\tau.
$$

Since $\mu_k \rightharpoonup \mu$ and $\zeta(\tau + \cdot) \in L^2(0, T - \tau)$ for $\tau \in (0, T)$, it holds $\int_0^{T-\tau} (\mu_k - \mu)(s) \zeta(\tau + s) ds \to 0$ for any $\tau \in (0, T)$. Moreover, since $\mu_k$ is bounded in $L^2(0, T)$, the sequence of $\tau$-dependent functions $\int_0^{T-\tau} (\mu_k - \mu)(s) \zeta(\tau + s) ds$ is bounded by a constant. In view of the Cauchy inequality and the dominated convergence theorem, we find

$$
|N_k| \leq \left\|v_k\right\|_{L^2(0,T)} \left\|\int_0^{T-\tau} (\mu_k - \mu)(s) \zeta(\cdot + s) ds\right\|_{L^2(0,T)} \to 0.
$$

Thus, from (3.77), due to $\mu_k \rightharpoonup \mu$, we obtain $\hat{\mu}_k \rightharpoonup \hat{\mu}$.

2. step. We estimate $J_3(z_k) - J_3(z)$ in terms of the difference of $\hat{u}_k$ and $\hat{u}$, where

$$
\hat{u} = u + \mu * u, \quad \hat{u}_k = u_k + \mu_k * u_k \quad \quad (3.78)
$$

and $u = u(x, t; z)$ and $u_k = u(x, t; z_k)$ are the weak solutions of (3.1) - (3.4) corresponding to the vectors $z$ and $z_k$, respectively.
The relations $u, u_k \in \mathcal{U}(Q)$ and $\mu, \mu_k \in L^2(0, T)$ imply $\hat{u}, \hat{u}_k \in \mathcal{U}(Q)$. Applying the operators $I - \hat{\mu}^*$ and $I - \hat{\mu}_k^*$ to the left and right equality in (3.78), respectively, and taking the relations $(I - \hat{\mu}^*)(I + \mu^*) = I$ and $(I - \hat{\mu}_k^*)(I + \mu_k^*) = I$ into account, we deduce the formulas

$$u = \hat{u} - \hat{\mu}^* \hat{u}, \quad u_k = \hat{u}_k - \hat{\mu}_k^* \hat{u}_k.$$

Subtracting we have

$$u_k - u = \hat{u}_k - \hat{\mu}_k^* (\hat{u}_k - \hat{u}) - (\hat{\mu}_k - \hat{\mu})^* \hat{u}.$$

Making use of the latter relation we express the difference of values of the functional $J_3$ as follows:

$$J_3(z_k) - J_3(z) = \int_c^d (u_k - u)^2(x, T)dx$$

$$+ 2 \int_c^d [u(x, T) - u_T(x)](u_k - u)(x, T)dx$$

$$+ \sum_{i=1}^{2} \int_0^T \left[ \sum_{l=1}^{L} \kappa_i(x_l, t)(u_k - u)(x_l, t) \right]^2 dt$$

$$+ \sum_{i=1}^{2} \int_0^T \left[ \sum_{l=1}^{L} \kappa_i(x_l, t)u(x_l, t) - v_i(t) \right] \left[ \sum_{l=1}^{L} \kappa_i(x_l, t)(u_k - u)(x_l, t) \right] dt$$

$$= I_1^1 + I_2^2 + I_3^3 + I_4^4,$$

where

$$I_1^1 = \int_c^d \left( \hat{u}_k - \hat{u} - \hat{\mu}_k^* (\hat{u}_k - \hat{u}) - (\hat{\mu}_k - \hat{\mu})^* \hat{u} \right)^2(x, T)dx,$$

$$I_2^2 = 2 \int_c^d [u(x, T) - u_T(x)]$$

$$\times (\hat{u}_k - \hat{u} - \hat{\mu}_k^* (\hat{u}_k - \hat{u}) - (\hat{\mu}_k - \hat{\mu})^* \hat{u} ) (x, T)dx,$$

$$I_3^3 = \sum_{i=1}^{2} \int_0^T \left[ \sum_{l=1}^{L} \kappa_i(x_l, t) \right.$$}

$$\times (\hat{u}_k - \hat{u} - \hat{\mu}_k^* (\hat{u}_k - \hat{u}) - (\hat{\mu}_k - \hat{\mu})^* \hat{u} ) (x_l, t) \left[ \sum_{l=1}^{L} \kappa_i(x_l, t)(u_k - u)(x_l, t) \right] dt,$$

$$I_4^4 = 2 \sum_{i=1}^{2} \int_0^T \left[ \sum_{l=1}^{L} \kappa_i(x_l, t)u(x_l, t) - v_i(t) \right]$$

$$\times \left[ \sum_{l=1}^{L} \kappa_i(x_l, t)(\hat{u}_k - \hat{u} - \hat{\mu}_k^* (\hat{u}_k - \hat{u}) - (\hat{\mu}_k - \hat{\mu})^* \hat{u} ) (x_l, t) \right] dt.$$
Let us estimate $I_1$. We split the ingredient of the integral up as $(\hat{u}_k - \hat{u} - \hat{\mu}_k * (\hat{u}_k - \hat{u}) - (\hat{\mu}_k - \hat{\mu}) * \hat{u})^2 = (\hat{u}_k - \hat{u} - \hat{\mu}_k * (\hat{u}_k - \hat{u}))^2 - 2(\hat{u}_k - \hat{u} - \hat{\mu}_k * (\hat{u}_k - \hat{u})) + (\hat{\mu}_k - \hat{\mu} * \hat{u})^2$. Using the Cauchy inequality, the boundedness of $L$ deduce

By the same reasons as above, it holds

|\begin{align*}
I_1^k &\leq \| (\hat{u}_k - \hat{u} - \hat{\mu}_k * (\hat{u}_k - \hat{u})) \|_{L^2(c,d)}^2 + 2\| (\hat{\mu}_k - \hat{\mu}) (\hat{u}_k - \hat{u}) \|_{L^2(c,d)} \| (\hat{u}_k - \hat{u} - \hat{\mu}_k * (\hat{u}_k - \hat{u})) \|_{L^2(c,d)} + R_k^1
\end{align*}|

where $\hat{C}_{12}$ is a constant and

$$R_k^1 = \int_c^d \left( \int_0^T (\hat{\mu}_k - \hat{\mu})(\tau) \hat{u}(x, T - \tau) d\tau \right)^2 dx.$$

Since $\hat{u} \in U(Q) \subset L^2(Q)$, by Tonelli’s theorem it holds $\hat{u}(x, \cdot) \in L^2(0, T)$ a.e. $x \in (c, d) \Rightarrow \hat{u}(x, T - \cdot) \in L^2(0, T)$ a.e. $x \in (c, d)$. Thus, in view of $\hat{\mu}_k \to \hat{\mu}$ in $L^2(0, T)$ we have $\int_0^T (\hat{\mu}_k - \hat{\mu})(\tau) \hat{u}(x, T - \tau) d\tau \to 0$ a.e. $x \in (c, d)$. Moreover, by the Cauchy inequality, the boundedness of $\hat{\mu}_k$ in $L^2(0, T)$ and $\hat{u} \in L^2(Q)$ we get

$$\int_0^T (\hat{\mu}_k - \hat{\mu})(\tau) \hat{u}(x, T - \tau) d\tau \leq \hat{C}_{13} \int_0^T (\hat{u}(x, \tau))^2 d\tau \in L^1(c, d)$$

with a constant $\hat{C}_{13}$. Therefore, due to the dominated convergence theorem we obtain $R_k^1 \to 0$.

Similarly, for $I_2^k$ we get

$$|I_2^k| \leq 2\| u(\cdot, T) - u_T \|_{L^2(c,d)} \| (\hat{u}_k - \hat{u} - \hat{\mu}_k * (\hat{u}_k - \hat{u})) (\cdot, T) \|_{L^2(c,d)} + R_k^2$$

where $\hat{C}_{14}$ is a constant and

$$R_k^2 = \int_c^d |u(x, T) - u_T(x)| \left| \int_0^T (\hat{\mu}_k - \hat{\mu})(\tau) \hat{u}(x, T - \tau) d\tau \right| dx.$$

By the same reasons as above, it holds $R_k^2 \to 0$.

Next, let us estimate $I_3^k$. Performing the same splitting as in $I_1^k$ we deduce

$$|I_3^k| \leq L^2 \sum_{i=1}^2 \max_{1 \leq l \leq L} \left[ \| \kappa_i(x_l, \cdot) \|_{L^\infty(0, T)} \| (\hat{u}_k - \hat{u} - \hat{\mu}_k * (\hat{u}_k - \hat{u})) (x_l, \cdot) \|_{L^2(0, T)}^2 \right]$$

$$+ 2L^2 \sum_{i=1}^2 \max_{1 \leq l \leq L} \left[ \| \kappa_i(x_l, \cdot) \|_{L^\infty(0, T)} \| (\hat{\mu}_k - \hat{\mu}) (\cdot) \|_{L^2(0, T)} \right]$$

$$\times \| (\hat{u}_k - \hat{u} - \hat{\mu}_k * (\hat{u}_k - \hat{u})) (x_l, \cdot) \|_{L^2(0, T)} + R_k^3$$

$$\leq \hat{C}_{15} \| (\hat{u}_k - \hat{u}) \|_{U(Q)}^2 + \| \hat{u}_k - \hat{u} \|_{U(Q)} + R_k^3,$$
where $\tilde{C}_{15}$ is a constant and
\[
R_k^3 = L^2 \sum_{i=1}^{2} \max_{1 \leq i \leq L} \left\{ \| \kappa_i(x_l, \cdot) \|_{L^\infty(0, T)}^2 \right\} \int_0^T \left[ \int_0^t \tilde{u}(x_l, t - \tau) d\tau \right]^2 dt.
\]
Since $\tilde{u}(x_l, t - \cdot) \in L^2(0, t) \forall t \in (0, T)$ we get $\int_0^t (\tilde{u}_k - \tilde{u})(\tau) \tilde{u}(x_l, t - \tau) d\tau \to 0$ \forall $t \in (0, T)$. Moreover, the sequence $|\int_0^t (\tilde{u}_k - \tilde{u})(\tau) \tilde{u}(x_l, t - \tau) d\tau|$ is bounded by a constant. Consequently, $R_k^3 \to 0$.

Finally, in an analogous manner we deduce the estimate for $I_k^4$:
\[
|I_k^4| \leq \tilde{C}_{16} \| \tilde{u}_k - \tilde{u} \|_{\mathcal{U}(Q)} + R_k^4, \quad \tilde{C}_{16} \text{ - a constant},
\]
\[
R_k^4 = 2L \sum_{i=1}^{2} \left[ \max_{1 \leq i \leq L} \left\{ \| \kappa_i(x_l, \cdot) \|_{L^\infty(0, T)} \right\} \int_0^T \left[ \int_0^t (\tilde{u}_k - \tilde{u})(\tau) \tilde{u}(x_l, t - \tau) d\tau \right]^2 dt \right]^{1/2},
\]
where $R_k^4 \to 0$.

In view of (3.79) and the deduced estimates of $I_k^1, \ldots, I_k^4$ it holds
\[
|J_3(z_k) - J_3(z)| \leq \tilde{C}_{17} (\| \tilde{u}_k - \tilde{u} \|_{\mathcal{U}(Q)} + \| \tilde{u}_k - \tilde{u} \|_{\mathcal{U}(Q)}) + R_k,
\]
where $\tilde{C}_{17}$ is a constant and $R_k = R_k^1 + \ldots + R_k^4 \to 0$.

3. Step. We prove that $\| \tilde{u}_k - \tilde{u} \|_{\mathcal{U}(Q)} \to 0$. This would imply $|J_3(z_k) - J_3(z)| \to 0$ and complete the proof.

The functions $\tilde{u}$ and $\tilde{u}_k$ are the weak solutions of the following problems (cp. the derivation of (3.30) in the proof of Theorem 3.1):
\[
\begin{align*}
\tilde{u}_t &= A\tilde{u} - \tilde{m} \ast A\tilde{u} + f + \phi_x \quad \text{in } Q, \\
\tilde{u} &= u_0 \quad \text{in } (c, d) \times \{0\}, \\
\tilde{u} &= \tilde{g} \quad \text{in } \Gamma_{1,T}, \\
- \nu_A \cdot \nabla \tilde{u} + \tilde{m} \ast \nu_A \cdot \nabla \tilde{u} &= h + \nu \cdot \phi \quad \text{in } \Gamma_{2,T},
\end{align*}
\]
\[
(3.80)
\]
\[
\begin{align*}
\tilde{u}_{k,t} &= A_k \tilde{u}_k - \tilde{m}_k \ast A_k \tilde{u}_k + f + \phi_x \quad \text{in } Q, \\
\tilde{u}_k &= u_0 \quad \text{in } (c, d) \times \{0\}, \\
\tilde{u}_k &= \tilde{g}_k \quad \text{in } \Gamma_{1,T}, \\
- \nu_A \cdot \nabla \tilde{u}_k + \tilde{m}_k \ast \nu_A \cdot \nabla \tilde{u}_k &= h + \nu \cdot \phi \quad \text{in } \Gamma_{2,T},
\end{align*}
\]
\[
(3.81)
\]
where $A_k v = (a_{11} v_x)_x + a_k v$ and
\[
\begin{align*}
\tilde{m} &= m + \tilde{\mu} - m \ast \tilde{\mu}, \quad \tilde{m}_k = m_k + \tilde{\mu}_k - m_k \ast \tilde{\mu}_k, \\
\tilde{g} &= g + \mu \ast g, \quad \tilde{g}_k = g + \mu_k \ast g.
\end{align*}
\]

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Let us show that \( \hat{m}_k \to \hat{m} \) in \( L^2(0,T) \). With an arbitrary \( \zeta \in L^2(0,T) \) we compute

\[
\langle \hat{m}_k - \hat{m}, \zeta \rangle_{L^2(0,T)} = \langle m_k - m, \zeta \rangle_{L^2(0,T)} + \langle \hat{\mu}_k - \hat{\mu}, \zeta \rangle_{L^2(0,T)} - N_k^1,
\]

where

\[
N_k^1 = \langle m_k \ast \hat{\mu}_k - m \ast \hat{\mu}, \zeta \rangle_{L^2(0,T)} = \int_0^T \hat{\mu}_k(\tau) \int_0^{T-\tau} (m_k - m)(s) \times \zeta(\tau + s) ds d\tau
\]

\[
+ \int_0^T m(\tau) \int_0^{T-\tau} (\hat{\mu}_k - \hat{\mu})(s) \zeta(\tau + s) ds d\tau.
\]

We use the relations \( m_k \to m, \hat{\mu}_k \to \hat{\mu} \) and treat the term \( N_k^1 \) similarly to the term \( N_k \) above to get \( N_k^1 \to 0 \). As a result we obtain \( \langle \hat{m}_k - \hat{m}, \zeta \rangle \to 0 \). This yields \( \hat{m}_k \to \hat{m} \).

Subtracting the problem of \( \hat{u} \) from the problem of \( \hat{u}_k \) we see that \( w_k := \hat{u}_k - \hat{u} \) is a weak solution of the following problem:

\[
\begin{align*}
w_{k,t} &= A w_k - \hat{m} \ast A w_k + \tilde{f}_k + \tilde{\phi}_{k,x} \quad \text{in } Q, \\
\hat{u} &= 0 & \text{in } (c,d) \times \{0\}, \\
\hat{u} &= \tilde{g}_k & \text{in } \Gamma_1, T, \\
- \nu_A \cdot \nabla w_k + \hat{m} \ast \nu_A \cdot \nabla w_k &= \nu \cdot \tilde{\phi}_k & \text{in } \Gamma_2, T,
\end{align*}
\]

where

\[
\tilde{f}_k = (a_k - a)(\hat{u}_k - \hat{m}_k \ast \hat{u}_k) - a(\hat{m}_k - \hat{m}) \ast \hat{u}_k,
\]

\[
\tilde{\phi}_k = - a_{11}(\hat{m}_k - \hat{m}) \ast \hat{u}_{k,x}, \quad \tilde{g}_k = (\mu_k - \mu) \ast g.
\]

In order to use the weak convergence \( a_k \to a \) in forthcoming estimations we have to introduce the functions \( \rho_k \in W^2_2(c,d) \) being the solutions of the following Neumann problems:

\[
\rho_k'' - \rho_k = a_k - a \quad \text{in } (c,d), \quad \rho'_k(c) = \rho'_k(d) = 0.
\]

Then \( \rho_k(x) = \int_c^d G(x,y)(a_k - a)(y) dy, \ x \in (c,d), \) where

\[
G(x,y) = \frac{1}{2(e^{c-d} - e^{d-c})} \left\{ \begin{array}{ll}
(e^{c-y} + e^{y-c})(e^{d-x} + e^{x-d}) & \text{for } y < x \\
(e^{c-x} + e^{x-c})(e^{d-y} + e^{y-d}) & \text{for } y > x
\end{array} \right.
\]

is a Green function that satisfies the properties \( G, G_x \in L^\infty((c,d)^2) \). The weak convergence \( a_k \to a \) in \( L^2(c,d) \) implies \(^1\)

\[
\|\rho_k\|_{W^2_2(c,d)} \to 0.
\]

\(^1\)This is the point we essentially use the assumption \( n = 1 \), because in case \( n \geq 2 \) the partial derivatives \( G_{x_i}(x,\cdot) \) are not elements of \( L^2(\Omega) \).
Using the relation \(a_k - a = \rho_k' - \rho\) we rewrite the term \((a_k - a)(\hat{u}_k - \hat{m}_k \ast \hat{u}_k)\) in \(f_k\) as follows:

\[
(a_k - a)(\hat{u}_k - \hat{m}_k \ast \hat{u}_k) = [\rho_k'(\hat{u}_k - \hat{m}_k \ast \hat{u}_k)]_x - \rho_k'(\hat{u}_k - \hat{m}_k \ast \hat{u}_k) - \rho_k(\hat{u}_k - \hat{m}_k \ast \hat{u}_k)
\]

and shift the addend \([\rho_k'(\hat{u}_k - \hat{m}_k \ast \hat{u}_k)]_x\) to the singular part \(\tilde{\phi}_{k,x}\). As a result, the problem for \(w_k\) is transformed to the form

\[
w_{k,t} = A w_k - \hat{m} \ast A w_k + \tilde{f}_k + \tilde{\phi}_{k,x} \quad \text{in } Q,
\]

\[\hat{u} = 0 \quad \text{in } (c, d) \times \{0\},\]

\[\hat{u} = \tilde{g}_k \quad \text{in } \Gamma_{1,T},\]

\[-v_A \cdot \nabla w_k + \hat{m} \ast v_A \cdot \nabla w_k = \nu \cdot \tilde{\phi}_k \quad \text{in } \Gamma_{2,T},\]

where

\[
\tilde{f}_k = -\rho_k'(\hat{u}_k + \hat{m}_k \ast \hat{u}_k)_x - \rho_k(\hat{u}_k + \hat{m}_k \ast \hat{u}_k) - a(\hat{m}_k - \hat{m}) \ast \hat{u}_k,
\]

\[
\tilde{\phi}_k = \rho_k'(\hat{u}_k + \hat{m}_k \ast \hat{u}_k) - a_11(\hat{m}_k - \hat{m}) \ast \hat{u}_k,x.
\]

Let \(t\) be an arbitrary number in \((0, T)\). As in the proofs of Theorems 2.1 and 3.1 we make use of the cutting operator \(P_t w = \begin{cases} w & \text{in } Q_t \\ 0 & \text{in } Q \setminus Q_t \end{cases} \). Let \(w_k^t\) stand for the weak solution of the problem (3.84) with \(\tilde{f}_k\) and \(\tilde{\phi}_k\) replaced by \(P_t \tilde{f}_k\) and \(P_t \tilde{\phi}_k\), respectively. Then, due to the causality \(w_k^t = w_k\) in \(Q_t\). Applying (3.20) to \(w_k^t\) we obtain

\[
\|w_k\|_{\mathcal{U}(Q_t)} = \|w_k^t\|_{\mathcal{U}(Q_t)} \leq \|w_k^t\|_{\mathcal{U}(Q)} \\
\leq \hat{C}_1 \left[\|P_t \tilde{f}_k\|_{L^2(0,t;L^1(c,d))} + \|P_t \tilde{\phi}_k\|_{L^2(Q)} + \theta_1 \|\tilde{g}_k\|_{\mathcal{T}(Q)}\right] \quad (3.85)
\]

\[
\|
\hat{f}_k\|_{L^2(0,t;L^1(c,d))} \leq \hat{C}_1 18 \left[\|\hat{m}_k - \hat{m}\|_{L^2(0,t;L^2(c,d))} + \|\rho_k\|_{W^1_2(c,d)}
\right. \\
\left. \times (1 + \|\hat{m}_k\|_{L^2(0,T)}) \|\hat{u}_k\|_{U(Q_t)} \right] \leq \hat{C}_1 18 \left[\int_0^t |\hat{m}_k - \hat{m}|(t - \tau) \|w_k\|_{L^2(Q_\tau)} d\tau \\
+ \|\rho_k\|_{W^1_2(c,d)} (1 + \|\hat{m}_k\|_{L^2(0,T)}) \|w_k\|_{U(Q_t)} \right] + R_k^t, \quad (3.86)
\]

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\[
\begin{align*}
\|\overline{\sigma}_k\|_{L^2(Q_t)} & \leq \hat{C}_{19} \left[ \|\tilde{m}_k - \hat{m}\|_{L^2(Q_t)} + \|\rho_k\|_{W^1_2(c,d)} (1 + \|\hat{m}_k\|_{L^2(0,T)}) \right] \\
& \times \|\hat{u}_k\|_{L^2(0,t;C[c,d])} \leq \hat{C}_{19} \left[ \int_0^t \|\tilde{m}_k - \hat{m}\|(t - \tau) \|w_k\|_{L^2(Q_\tau)} d\tau \\
& + \|\rho_k\|_{W^1_2(c,d)} (1 + \|\hat{m}_k\|_{L^2(0,T)}) \|w_k\|_{L^2(0,t;C[c,d])} \right] + \overline{R}_k^2, \\
\|\hat{g}_k\|_{U(Q)} & \leq \overline{R}_k^3,
\end{align*}
\]

where

\[
\begin{align*}
\overline{R}_k^1 & = \hat{C}_{18} \left[ \|\tilde{m}_k - \hat{m}\|_{L^2(Q)} + \|\rho_k\|_{W^1_2(c,d)} (1 + \|\hat{m}_k\|_{L^2(0,T)}) \|\hat{u}\|_{U(Q)} \right], \\
\overline{R}_k^2 & = \hat{C}_{19} \left[ \|\tilde{m}_k - \hat{m}\|_{L^2(Q)} + \|\rho_k\|_{W^1_2(c,d)} (1 + \|\hat{m}_k\|_{L^2(0,T)}) \|\hat{u}\|_{L^2(0,T;C[c,d])} \right], \\
\overline{R}_k^3 & = \|\mu_k - \mu\|_{L^2(Q)} + \|\mu_k - \mu\|_{L^2(Q)} + \|\mu_k - \mu\|_{L^2(Q)}
\end{align*}
\]

and \( \hat{C}_{18}, \hat{C}_{19} \) are constants. The quantities \( \overline{R}_k^j, j = 1, 2, 3 \) contain terms with the factor \( \|\rho_k\|_{W^1_2(c,d)} \) and terms of the form \( \|\hat{u}_k\|_{L^2(Q)} \), where \( z_k \) is one of the functions \( \hat{m}_k - \hat{m}, \mu_k - \mu \) or \( \hat{m}_k - \hat{m} \) and \( \hat{v} \in L^2(Q) \) is one of the functions \( \hat{u}_x, \hat{u}_x, g, g_x \) or \( g_t \). The former terms converge to zero because of the relation (3.83) and the boundedness of \( \hat{m}_k \) in \( L^2(0,T) \) and the latter terms approach zero by virtue of the weak convergence \( z_k \to 0 \) in \( L^2(0,T) \). More precisely, to prove that \( \|z_k \ast \hat{v}\|_{L^2(Q)} \to 0 \) we write

\[
\|z_k \ast \hat{v}\|_{L^2(Q)} = \left\{ \int_0^T \int_c^d \left[ \int_0^t z_k(\tau) \hat{v}(x, t - \tau) d\tau \right]^2 dx dt \right\}^{1/2}.
\]

The component \( \left[ \int_0^t z_k(\tau) \hat{v}(x, t - \tau) d\tau \right]^2 \) is bounded by an integrable with respect to \( x \) in \((c,d)\) function \( \sup_k \|z_k\|_{L^2(0,T)}^2 \|\hat{v}(x, \cdot)\|_{L^2(0,T)}^2 \) and tends to zero for all \( t \in (0,T) \) and a.e. \( x \in (c,d) \), because \( z_k \to 0 \) and \( \hat{v}(x, t - \cdot) \in L^2(0,T) \) for all \( t \in (0,T) \) and a.e. \( x \in (c,d) \). (The latter relation follows from \( \hat{v} \in L^2(Q) \) and Tonelli’s theorem.) Thus, by the dominated convergence theorem, it holds \( \|z_k \ast \hat{v}\|_{L^2(Q)} \to 0 \). Summing up,

\[
\overline{R}_k^j \to 0, \quad j = 1, 2, 3.
\]

As in proof of Theorem 3.1, we use the norms \( \|w\|_{\sigma} = \sup_{0 < t < T} e^{-\sigma t} \|w\|_{U(Q_t)} \) with the weights \( \sigma \geq 0 \) in the space \( U(Q) \). Then in view of (3.86) - (3.88)
from (3.85) we deduce

\[
\|w_k\|_\sigma \leq \hat{C}_{20} \left[ \sup_{0 < t < T} \int_0^t e^{-\sigma(t-\tau)}|\hat{m}_k - \hat{m}|(t-\tau)e^{-\sigma\tau}\|w_k\|_{U(Q,\tau)}d\tau \\
+ \|\rho_k\|_{W_2^1(c,d)}(1 + \|\hat{m}_k\|_{L^2(0,T)})\|w_k\|_\sigma + 3 \sum_{j=1}^3 R_j^k \right] \\
\leq \hat{C}_{21} \left[ \left\{ \|e^{-\sigma t}\|_{L^2(0,T)}\|\hat{m}_k - \hat{m}\|_{L^2(0,T)} \right. \\
+ \left. \|\rho_k\|_{W_2^1(c,d)}(1 + \|\hat{m}_k\|_{L^2(0,T)}) \right\} \|w_k\|_\sigma + 3 \sum_{j=1}^3 R_j^k \right],
\]

(3.90)

where \( \hat{C}_{20}, \hat{C}_{21} \) are constants. Since \( \|e^{-\sigma t}\|_{L^2(0,T)} \to 0 \) as \( \sigma \to \infty \), \( \|\rho_k\|_{W_2^1(c,d)} \to 0 \) and the sequence \( \|\hat{m}_k\|_{L^2(0,T)} \) is bounded, there exist \( \sigma > 0 \) and \( K_2 \in \mathbb{N} \) such that

\[
\hat{C}_{21} \left\{ \|e^{-\sigma t}\|_{L^2(0,T)}\|\hat{m}_k - \hat{m}\|_{L^2(0,T)} + \|\rho_k\|_{W_2^1(c,d)}(1 + \|\hat{m}_k\|_{L^2(0,T)}) \right\} \leq \frac{1}{2}
\]

for \( k \geq K_2 \). This with (3.90) implies

\[
\|w_k\|_\sigma \leq 2\hat{C}_{21} \sum_{j=1}^3 R_j^k \quad \text{and hence} \quad \|w_k\|_{U(Q)} \leq 2e^{\sigma T}\hat{C}_{21} \sum_{j=1}^3 R_j^k
\]

for \( k \geq K_2 \). Taking (3.89) into account we obtain the desired convergence \( \|\hat{u}_k - \hat{u}\|_{U(Q)} = \|w_k\|_{U(Q)} \to 0 \). The theorem is proved. ■

3.4 Discretization and minimization

In the final section of the thesis let us discuss some aspects of the discretization and minimization of the cost functionals by means of the penalized gradient method.

Let us consider any of the problems IP4, IP5 or IP6 and search for its quasi-solution. This means that we have to minimize corresponding cost functional \( J_k \) over a given set \( M \subseteq \mathcal{Z}_k \), where \( k \) is some number in the set \( \{1; 2; 3\} \). In order to treat these problems in a common manner, we use the notation \( z \) also for the solution \( \omega \) of IP4.

Clearly, there are many possibilities to discretize IP4 - IP6. Here we describe in detail an orthogonal discretization. Let us introduce a \( L \)-dimensional subspace \( \mathcal{Z}_{k,L} \) of \( \mathcal{Z}_k \) (here \( L < \infty \)). Let \( P_L \) stand for the orthogonal projection to \( \mathcal{Z}_{k,L} \). Then the \( L \)-dimensional analogue of the set \( M \) is

\[
M_L = P_L M \subseteq \mathcal{Z}_{k,L}.
\]
Now we replace the problem of minimization of \( J_k \) over \( M \) by the following penalized discrete problem: find \( z^\dagger \) such that

\[
z^\dagger \in \arg \min_{z \in \mathbb{Z}_{k,L}} \Phi_{k,L}(z), \quad \text{where} \quad \Phi_{k,L} = \Pi_L + J_k \tag{3.91}
\]

and \( \Pi_L \) is a penalty function corresponding to the set \( M_L \). In order the problem to be relevant, \( \Pi_L \) has to be small inside \( M_L \) and large outside \( M_L \). The mathematical conditions imposed on \( \Pi_L \) are

\( \Pi_L \) is coercive, convex, Fréchet differentiable,

The Fréchet derivative \( \Pi_L' \) of \( \Pi_L \) is uniformly Lipschitz continuous in \( \mathbb{Z}_{k,L}^* \).

\( \Phi_{k,L} = \Pi_L + J_k \).

**Theorem 3.11** Let the assumptions listed in §3.2.1 (k) be satisfied. Moreover, let \( (3.92) \) hold. Then the problem \( (3.91) \) has a solution.

**Proof.** The proof is similar to the proof of Theorem 3.8. By coercitivity and continuity, \( \Pi_L \) is bounded from below. Moreover, \( J_k \) is also bounded from below. Thus, there exists \( m = \inf_{z \in \mathbb{Z}_{k,L}} \left[ \Pi_L(z) + J_k(z) \right] = \inf_{z \in \mathbb{Z}_{k,L}} \Phi_{k,L}(z) > -\infty \). Let \( z_l \in \mathbb{Z}_{k,L} \) be a minimizing sequence, i.e. \( \lim_{l \to \infty} \Phi_{k,L}(z_l) = m \). Due to the coercitivity of \( \Pi_L \), the sequence \( z_l \) is bounded (in case \( z_l \) is not bounded, there is a subsequence \( z_{l_i} \) such that \( \| z_{l_i} \| \to \infty \), hence by the coercitivity \( \Phi_{k,L}(z_{l_i}) \to \infty \), but this is in contradiction with the relation \( \lim_{l \to \infty} \Phi_{k,L}(z_l) = m \)). In a finite-dimensional space every bounded sequence is compact. Consequently, there exists a subsequence \( z_{l_j} \) such that \( \lim_{j \to \infty} z_{l_j} = z^* \). Due to the continuity of \( \Phi_{k,L} \), following from the Fréchet differentiability, we have \( \lim_{j \to \infty} \Phi_{k,L}(z_{l_j}) = \Phi_{k,L}(z^*) \). Thus, \( \Phi_{k,L}(z^*) = m \) and \( z^* \in \arg \min_{z \in \mathbb{Z}_{k,L}} \Phi_{k,L}(z) \). The element \( z^* \) is a solution of (3.91).

Further, we formulate an algorithm of the gradient method for the minimization of \( \Phi_{k,L} \). To this end, the representations of Fréchet derivatives of \( J_k \) obtained in Theorems 3.3 - 3.7 are useful. According to these theorems, \( J'_k(z) \Delta z = \langle \varrho_k, \Delta z \rangle_{\mathbb{Z}_k} \), where the \( z \)-dependent element \( \varrho_k = \varrho_k[z] \in \mathbb{Z}_k \) is given by (3.57), (3.43) and (3.69) - (3.72) in cases \( k = 1, 2 \) and \( 3 \), respectively. In order to construct an algorithm that remains inside the subspace \( \mathbb{Z}_{k,L} \), we have to find an analogue of \( \varrho_k \) in \( \mathbb{Z}_{k,L} \). This is \( P_L \varrho_k \). Indeed, for any \( \Delta z \in \mathbb{Z}_{k,L} \) we have

\[
\langle \varrho_k, \Delta z \rangle_{\mathbb{Z}_k} = \langle P_L \varrho_k, \Delta z \rangle_{\mathbb{Z}_k} + \langle \varrho_k - P_L \varrho_k, \Delta z \rangle_{\mathbb{Z}_k} = \langle P_L \varrho_k, \Delta z \rangle_{\mathbb{Z}_k},
\]

because \( \varrho_k - P_L \varrho_k \perp \Delta z = 0 \) since \( P_L \) is the orthogonal projection onto \( \mathbb{Z}_{k,L} \). Thus, \( P_L \varrho_k \) can be used as a representative of Fréchet derivative of \( J_k \) in \( \mathbb{Z}_{k,L} \).
Let the representative of $\Pi'_L(z)$ in $Z_{k,L}$ be $\pi_L[z]$, i.e. $\Pi'_L(z)\Delta z = \langle \pi_L[z], \Delta z \rangle_{Z_k}$ for $\Delta z \in Z_{k,L}$. Then the Fréchet derivative of $\Phi_{k,L}$ at $z$ has the formula

$$\Phi'_{k,L}(z)\Delta z = \langle G_k[z], \Delta z \rangle_{Z_k}$$

for $\Delta z \in Z_{k,L}$, where $G_k[z] = \pi_L[z] + P_L g_k[z]$.

The gradient method is as follows. We choose some initial guess $z_0 \in Z_{k,L}$ and compute successive approximate solutions by means of the formula

$$z_{s+1} = z_s - c_s G_k[z_s], \quad (3.93)$$

where $s = 0, 1, 2, \ldots$ and $c_s > 0$.

**Theorem 3.12** Let $k \in \{1; 2\}$, i.e. we have either IP4 or IP5. Assume that the assumptions listed in §3.2.1 (k) are satisfied and (3.92) hold. Moreover, let $c_s$ be chosen by the rule

$$\inf_{c > 0} \Phi_{k,L}(z_0 - cG_k[z_0]) \leq \Phi_{k,L}(z_0 - c_s G_k[z_s]) \leq \inf_{c > 0} \Phi_{k,L}(z_0 - cG_k[z_0]) + \delta_s,$$

where $\delta_s \geq 0$, $\sum_{s=0}^{\infty} \delta_s =: \delta < \infty$. Then it holds $\operatorname{dist}(z_s, S) \to 0$ as $s \to \infty$, where $S$ is the set of solutions of (3.91).

**Proof.** Let us prove the theorem in case $k = 1$ (IP4). The proof in case $k = 2$ is similar.

The assertion follows from Theorem 5.1.2 of [63] once we have proved that $G_1$ is uniformly Lipschitz-continuous with respect to $z$, the functional $\Phi_{1,L}$ is convex and the set $M(z_0) = \{z \in Z_{1,L} : \Phi_{1,L}(z) \leq \Phi_{1,L}(z_0) + \delta\}$ is bounded. The convexity of $\Phi_{1,L}$ follows from the convexity of its addends $\Pi_L$ and $J_1$ (the latter one was shown in the proof of Theorem 3.9). The boundedness of $M(z_0)$ is a direct consequence of the coercitivity of $\Phi_{1,L}$ following from the coercitivity of the addend $\Pi_L$.

It remains to show the uniform Lipschitz continuity of $G_1[z]$ with respect to $z$ in $Z_{1,L}$. The Lipschitz continuity of $\pi_L[z]$ in $Z_{1,L}$ follows from the assumed Lipschitz continuity of $\Pi'_L[z]$ in $Z_{1,L}^*$, because $Z_{1,L}$ and $Z_{1,L}^*$ are isomorphic and $\pi_L[z]$ is the analogue of $\Pi'_L[z]$ in $Z_{1,L}$. Thus, we have still to prove the Lipschitz continuity of the term $P_L g_1[z]$ with respect to $z$ occurring in the formula of $G_1[z]$.
Using (3.50) we obtain for any \( z, \tilde{z} \in Z_{1,L} \)

\[
\|P_L \varrho_1[\tilde{z}] - P_L \varrho_1[z]\|_{z_1} \leq \|P_L\| \|\varrho_1[\tilde{z}] - \varrho_1[z]\|_{z_1}
\]

\[
\leq C_{22} \sum_{i=1}^{N} \|\psi_i(\cdot, \cdot; \tilde{z}) - \psi_i(\cdot, \cdot; z)\|_{L^2(Q)}
\]

\[
\leq C_{23} \sum_{i=1}^{N} \|\psi_i(\cdot, \cdot; \tilde{z}) - \psi_i(\cdot, \cdot; z)\|_{U(Q)},
\]

where the constants \( C_{22} \) and \( C_{23} \) are independent of \( z \) and \( \tilde{z} \). The functions \( \psi_i(\cdot, \cdot; \tilde{z}) \) and \( \psi_i(\cdot, \cdot; z) \) solve the problems (3.51) with \( \omega = \tilde{z} \) and \( \omega = z \), respectively. Applying the relation (3.20) of Theorem 3.1 to the problem for the difference \( \psi_i(\cdot, \cdot; \tilde{z}) - \psi_i(\cdot, \cdot; z) \) we continue the estimate as follows:

\[
\|P_L \varrho_1[\tilde{z}] - P_L \varrho_1[z]\|_{z_1} \leq 2C_{23} \hat{C}_1 \sum_{i=1}^{N} \|u(\cdot, T_i; \tilde{z}) - u(\cdot, T_i; z)\|_{L^2(\Omega)}
\]

\[
\leq C_{24} \|u(\cdot, \cdot; \tilde{z}) - u(\cdot, \cdot; z)\|_{U(Q)}
\]

with a constant \( C_{24} \) independent of \( z \) and \( \tilde{z} \). The function \( u(\cdot, \cdot; \tilde{z}) - u(\cdot, \cdot; z) \) is a weak solution of the problem (3.1) - (3.4) with \( f(x, t) = \sum_{j=1}^{N} \gamma_j(t)[\tilde{z}_j(x) - z_j(x)] \), \( \phi = 0, \varphi = 0, u_0 = 0, g = 0, h = 0 \). Using again (3.20) we obtain

\[
\|P_L \varrho_1[\tilde{z}] - P_L \varrho_1[z]\|_{z_1} \leq 2C_{24} \hat{C}_1 \left\| \sum_{j=1}^{N} \gamma_j[\tilde{z}_j - z_j] \right\|_{L^2(0,T;L^{q_2}(\Omega))}
\]

\[
\leq C_{25} \|\tilde{z} - z\|_{z_1}
\]

with a constant \( C_{25} \) independent of \( z \) and \( \tilde{z} \). This proves the uniform Lipschitz-continuity of \( P_L \varrho_1 \). \( \square \)

The convergence of \( z_s \) in case \( k = 3 \) is an open issue. This case is more complex because IP6 is nonlinear and the Fréchet derivative of \( J_3 \) is not uniformly Lipschitz-continuous.

The quasi-solutions of IP4 - IP6 are not expected to be stable with respect to noise of the data, i.e. the problems under consideration may be ill-posed. Nevertheless, from the intuitive viewpoint a discretisation should regularize an ill-posed problem. Such a property of the discretization has been proved in many cases [40, 56]. Alternatively, the index \( s \) of the gradient method could be used as a regularization parameter (cf. [19]). Moreover, the addend \( \Pi_L \) can be defined to be the stabilizing term of the Tikhonov’s method instead of the penalty function, i.e. \( \Pi_L = \alpha \|\tilde{z}\|^2 \), where \( \alpha > 0 \) is the regularization parameter. Such a \( \Pi_L \) satisfies (3.92).
REFERENCES


Acknowledgements

I would like to thank my supervisor Prof. Jaan Janno. I am deeply grateful for his time and patience, his guidance and understanding.
ABSTRACT


The aim of the thesis is to perform a systematical study of inverse problems for parabolic integro-differential equations containing time convolutions with memory kernels in case space-dependent factors of free terms or coefficients of the equations are unknown and observation conditions are given either in the form of instant measurements over the space or integrated with respect to time measurements over the space. In the second part of the thesis a problem to determine kernels is also considered.

First part of the thesis contains an analysis of problems that are smooth in the sense that all derivatives included in the integro-differential equations are regular functions. A positivity principle for parabolic integro-differential equations is established. In case final data for a solution of a direct problem are given, the global existence, uniqueness and stability for an inverse problem to determine a space-dependent component of a free term of the equation are proved. The proof of the uniqueness uses the positivity principle and the proof of existence and stability exploits the Fredholm alternative. Moreover, making use of results obtained for the inverse free term problem, global uniqueness, local existence and stability for inverse problems to determine a lower-order coefficient and a coefficient at the time derivative occurring in the equation from final data are proved. The main tool is the Banach fixed-point principle.

Second part of the thesis includes a treatment of problems in non-smooth case, i.e. when higher order derivatives involved in an integro-differential equation are singular distributions. A weak convolutional form of the direct problem is introduced. Such a form does not involve time derivatives of solutions or test functions. A general method to derive adjoint problems for Fréchet derivatives of cost functionals corresponding to inverse problems for this equation is proposed. The method makes use of the mentioned weak convolutional form. Further, this method is applied to deduce adjoint problems for 3 particular inverse problems: a problem to determine a finite number of space-dependent factors of a free term from instant measurements; a problem to determine a finite number of space-dependent factors of a free term and an initial condition from integral measurements; a problem to determine a lower-order coefficient and two kernels from combined instant and integral measurements. The existence of quasi-solutions for the mentioned 3 inverse problems is established. The existence result for the third inverse problem (that is nonlinear) in a non-compact set is proved in the one-dimensional case.
KOKKUVÕTE


Doktoritöö eesmärgiks on süstemaatiliselt analüüsida pöördülesandeid mälutuumadega konvolutsiooniliikmeid sisaldavatele paraboolsetele integrodifferentsiaalvõrranditele juhul, kui tundmatuteks on vabaliikmete ruumimuutujatest sõltuvad tegurid või kordajad ja vaatlused on antud kas hetkingimustena üle ruumi või integreeritud mõõtmistena aja suhtes üle ruumi. Töö teises osas on vaatluse all ka tuumade määramise ülesanne.


Curriculum Vitae

1. Personal data

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2. Education

<table>
<thead>
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3. Language competence (fluent, average, basic skills)

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4. Professional employment

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<td>2003-2005</td>
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<tr>
<td>1999-2002</td>
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5. Scientific work

4. K. Kasemets, J. Janno, Inverse problems for parabolic integro-differential
equations with two kernels. *18th International Conference Mathematical Modelling and Analysis and 4th International Conference Approximation Methods and Orthogonal Expansions, Tartu, 27-30.05.2013.* p 57.


6. Defended thesis


7. Main areas of scientific work

Differential Equations, Inverse Problems, Ill-posed problems, Integral Equations.
Elulookirjeldus

1. Isikuantmed

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2. Hariduskäik

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<td>1999-2002</td>
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5. Teadustegevus

3. K. Kasemets, J. Janno, Inverse problems for parabolic integro-differential...


6. Kaitstud lõputöö


7. Teadustöö põhisuunad

Diferentsiaal- ja integraalvõrrandid, pöördülesanded, mittekorrektsed ülesanded
Appendix
Publication I

A POSITIVITY PRINCIPLE FOR PARABOLIC
INTEGRO-DIFFERENTIAL EQUATIONS AND INVERSE
PROBLEMS WITH FINAL OVERDETERMINATION

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(Communicated by Victor Isakov)

Abstract. A positivity principle for parabolic integro-differential equations is proved. By means of this principle, uniqueness, existence and stability for an inverse source problem and two inverse coefficient problems are established.

1. Introduction. Inverse problems for parabolic differential equations with final overdetermination have been studied by many authors [1, 2, 6, 7, 8, 12]. The present paper represents an attempt to generalize some of these results to the case of parabolic integro-differential equations. More precisely, we will be limited to inverse problems for parabolic equations in the diffusion processes with memory that involve time convolutions of a kernel \( m \) and an elliptic operator \( A \) of the concentration or temperature \( u \) (equation (1) below).

A linear inverse problem to determine a source term in a parabolic integro-differential equation by means of the final overdetermination was previously studied in [11]. More precisely, in this paper the authors prove the existence and uniqueness for the inverse problem under the assumption that the solution is unique. The proof is based on Fredholm alternative.

The uniqueness is a more difficult issue. Its study requires a suitable positivity principle for mentioned parabolic integro-differential equations. To the authors’ knowledge, such a positivity result has not been known in the literature. Our idea to prove this principle is as follows. We transform the equation to a form that contains a time convolution of the resolvent \( k \) of \( m \) and the time derivative of \( u \) instead of \( Au \). Then it is possible to integrate by parts in this convolution to remove completely the derivatives of \( u \). Owing to this possibility, we can use the classical proof scheme of extremum principles. Handling the additional convolution term is not very difficult, because we can operate with the sign of \( u \) instead of \( Au \) (see part 1 of the proof of Theorem 2). We prove the positivity principle in the cases of first and second kind boundary conditions under the assumptions that the resolvent kernel

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satisfies the conditions $k \geq 0$ and $k_i \leq 0$. The problem with third kind boundary conditions cannot be handled by the technique presented in the paper.

It turns out that the presence of a non-vanishing kernel $k$ with such properties weakens other assumptions of the positivity principle. For instance, for the source term $\chi$, it is sufficient to assume $\chi + k \ast \chi \geq 0$. This assumption is weaker than the assumption $\chi \geq 0$ that occurs in the case of usual parabolic differential equation. Physically this is explained by an inertia of the medium. The same remark holds for the second kind boundary condition, too. Details are given in Remark 2 below.

The analysis of inverse problems in this paper uses very much the technique of paper [7]. In particular, the decomposition of $z$ into $z^+$ and $z^-$ in Theorem 3 comes from [7] and proof of Theorem 4 uses an analogous theorem from [7].

In this paper we prove existence, uniqueness and stability theorems for the linear inverse source problem and two nonlinear inverse coefficient problems for parabolic integro-differential equations. Existence and stability results for coefficient problems are local. Again, the assumptions are in case $k \neq 0$ weaker than in the case $k = 0$.

The method of the paper works only in case the memory kernel $m$ is independent of spatial variables. Otherwise it falls into a divergence-nabla type operator and a time resolvent is not applicable.

2. Formulation of problems. Notation. Linear non-homogeneous diffusion processes with memory are subject to the constitutive relation [3, 4, 9, 13]

$$q_i(x,t) = -\sum_{j=1}^{n} a_{ij}(x)u_{x_j}(x,t) + \int_{0}^{t} m(t-\tau) \sum_{j=1}^{n} a_{ij}(x)u_{x_j}(x,\tau)d\tau, \quad i = 1, \ldots, n,$$

where $q$ is the flux and $u$ is the concentration (or temperature). The function $m$ is the memory kernel. We will consider the case when $m$ is independent of $x$.

Using this relation in the continuity equation $\beta u_t + \text{div} q = \chi$ which is assumed to be valid in some open domain $(x,t) \in Q = \Omega \times (0,T) \subset \mathbb{R}^{n+1}$, where $\beta : \Omega \rightarrow \mathbb{R}$ and $\chi : Q \rightarrow \mathbb{R}$ are given functions, we arrive at the parabolic integro-differential equation

$$(1) \quad \beta u_t = Au - m \ast Au + \chi \quad \text{in} \quad Q,$$

where $A = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial}{\partial x_j} \right)$ and $\ast$ stands for the time convolution, i.e.

$$v \ast w = \int_{0}^{t} v(\cdot - \tau)w(\tau)d\tau.$$

In this paper we generalize a bit this physical model. Namely, we consider the equation (1) with the operator $A$ of the form

$$(2) \quad A = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} a_{ij}(x)\frac{\partial}{\partial x_j} + a(x,t)$$

where $a_{ij}$, $a_j$, and $a$ are given coefficients. We assume that the principal part of $A$ is positive definite, i.e. the inequality

$$(3) \quad \sum_{i,j=1}^{N} a_{ij} \lambda_i \lambda_j \geq \epsilon |\lambda|^2 \quad \text{for any} \quad x \in \Omega, \quad \lambda \in \mathbb{R}^n \quad \text{and some} \quad \epsilon \in (0, \infty)$$
holds. Since the operator $A$ depends on $t$, we will write $A = A(t)$, if necessary. Moreover, let the relation

$$(4) \quad \beta \geq \beta_0 > 0 \quad \text{for any } x \in \Omega \quad \text{and some } \beta_0 \in (0, \infty)$$

be valid.

We supplement the equation (1) with initial and boundary conditions

$$(5) \quad u = u_0 \quad \text{in } \Omega \times \{0\}, \quad Bu = b \quad \text{in } S = \Gamma \times (0, T)$$

where $u_0$, $b$ are some given functions, $\Gamma$ is the boundary of $\Omega$ and either

$$(6) \quad Bu = u \quad \text{(case I)}$$

or

$$(7) \quad Bu(x, t) = \omega(x) \cdot \nabla_x u(x, t) - \int_0^t m(t - \tau) \omega(x) \cdot \nabla_x u(x, \tau) d\tau \quad \text{(case II)}$$

with $\omega(x) \cdot N(x) > 0$, $N(x)$ - outer normal of $\Gamma$ at $x$. Note that case II is a generalization of the second kind boundary condition $-q \cdot N = b$. In the sequel we denote by $\nu$ the order of the boundary operator $B$, i.e $\nu = 0$ in case I and $\nu = 1$ in case II.

Throughout the paper we assume the boundary $\Gamma$ and the function $\omega$ sufficiently smooth.

The relations (1) and (5) form the direct problem for $u$. In this paper we will study three following inverse problems:

**IP1:** Let the free term be of the following form:

$$(8) \quad \chi(x, t) = z(x) \phi(x, t) + \chi_0(x, t).$$

Given $m, \beta, a_0, a_1, a_2, b, \phi, \chi_0$ and a function $u_T(x)$, $x \in \Omega$, find $z$ and $u$ so that the relations (1), (5), (8) and

$$u = u_T \quad \text{in } \Omega \times \{T\}$$

hold.

**IP2:** Let $a_1 = 0$. Given $m, \beta, a_0, a_2, b, \chi$ and a function $u_T(x)$, $x \in \Omega$, find $a$ and $u$ so that the relations (1), (5) and (9) hold.

**IP3:** Given $m, a_0, a_1, a_2, b, \chi$ and a function $u_T(x)$, $x \in \Omega$, find $\beta$ and $u$ so that the relations (1), (5) and (9) hold.

Let us introduce the functional spaces we will use in the paper. Given the Banach space $X$, $p \in [1, \infty)$ and an integer $l \geq 0$, we define the following abstract Sobolev spaces:

$$(9) \quad W^l_p(0, T; X) = \left\{ v \in [0, T] \to X : \|v\|_{W^l_p(0, T; X)} := \sum_{j=0}^l \left( \int_0^T \|v^{(j)}(t)\|_X^p \, dx \right)^{\frac{1}{p}} < \infty \right\}.$$ 

In case $l = 0$ they coincide with the abstract Lebesgue spaces, i.e. $W^0_p(0, T; X) = L^p(0, T; X)$. If $X = \mathbb{R}$, we merely write $W^l_p(0, T; \mathbb{R}) = W^l_p(0, T)$. In addition, we need spaces of fractional order and anisotropic spaces. To this end, let us first introduce the following notation for difference quotients of $x$- and $(x, t)$-dependent functions with powers:

$$(10) \quad \langle v \rangle_p(x_1, x_2) := \frac{v(x_1) - v(x_2)}{|x_1 - x_2|^p}, \quad \langle v \rangle_p(x_1, x_2; t) := \frac{v(x_1, t) - v(x_2, t)}{|x_1 - x_2|^p},$$

$$(11) \quad \langle v \rangle_p(x; t_1, t_2) := \frac{v(x, t_1) - v(x, t_2)}{|t_1 - t_2|^p}.$$
For any $p \in [1, \infty)$ and $l \geq 0$ we define the Sobolev-Slobodeckij spaces (cf. [10, 14])

$$W^l_p(\Omega) = \left\{ v : \|v\|_{W^l_p(\Omega)} := \sum_{|\alpha| \leq l} \left[ \int_\Omega |D^\alpha v(x)|^p \, dx \right]^\frac{1}{p} + \Theta_l \sum_{|\alpha| = l} \left[ \int_\Omega dx_1 \int_\Omega |(D^\alpha v(x_1, x_2)|^p \, dx \right]^\frac{1}{p} < \infty \right\},$$

$$W^l_\infty(Q) = \left\{ v : \|v\|_{W^l_\infty(Q)} := \sum_{2j + |\alpha| \leq l} \left[ \int_0^T dt \int_\Omega dx \int_\Omega |(D^\alpha D^\alpha v(x, t)|^p \, dx \, dt \right]^\frac{1}{p} + \Theta_l \sum_{0 < l - |\alpha| < 2l} \left[ \int_\Omega dx \int_0^T dt \int_0^T dt_1 |(D^\alpha D^\alpha v(\cdot, \cdot, \cdot)|^p \, dt_1 \, dt_2 \right]^\frac{1}{p} < \infty \right\}.$$

Here $[l]$ is the greatest integer $\leq l$ and $\Theta_l = 0$ and $\Theta_l = 1$ in the cases of integer $l$ and non-integer $l$, respectively. Further, for any non-integer $l > 0$ we define the Hölder spaces

$$C^l(\Omega) = \left\{ v : D^\alpha v \in C(\Omega) \text{ for } |\alpha| \leq l, \|v\|_l := \sum_{|\alpha| \leq l} \left[ \sup_{x \in \Omega} |D^\alpha v(x)| + \sup_{x_1, x_2 \in \Omega} |(D^\alpha v)(x_1, x_2)| \right] < \infty \right\},$$

$$C^{2j}_\infty(Q) = \left\{ v : D^\alpha D^\alpha v \in C(Q) \text{ for } 2j + |\alpha| \leq l, \|v\|_{2j} := \sum_{2j + |\alpha| \leq l} \left[ \sup_{(x, t) \in Q} \left| D^\alpha D^\alpha v(x, t) \right| + \sup_{x_1, x_2 \in \Omega} \left| (D^\alpha D^\alpha v)(x_1, x_2, t) \right| \right] < \infty \right\}.$$

The definitions of $W^l_\infty$ and $C^{2j}_\infty$ are in a standard manner extended from $Q$ to the boundary cylinder $S$ (for details see [10]). For integer $l \geq 0$ we define

$$C^{2l}(\overline{Q}) = \left\{ v : D^\alpha D^\alpha v \in C(\overline{Q}) \text{ for } 2j + |\alpha| \leq 2l \right\}.$$

Finally, we introduce some additional notation. Let $U$ be a finite-dimensional manifold and $f, g \in L^1(U)$. We write

$$f \geq g \text{ in } U \quad \text{if } f(x) \geq g(x) \text{ a.e. } x \in U,$$

$$f > g \text{ in } U \quad \text{if } \forall U_1 : \overline{U_1} \subseteq U \exists \varepsilon_{U_1} \in \mathbb{R}, \varepsilon_{U_1} > 0 : f \geq g + \varepsilon_{U_1} \text{ in } U_1.$$

It is not difficult to prove that

$$f \geq g, f \neq g \text{ in } U \quad \Rightarrow \quad \exists U_2 \subseteq U : \text{meas } U_2 \neq 0, f > g \text{ in } U_2.$$

3. Results for direct problem. Positivity principle. The technique of the paper is based on the representation of the memory kernel $m$ in terms of its resolvent $k$. Namely, we define $k$ to be a solution of the following Volterra integral equation:

$$k(t) = \int_0^t m(t - \tau) k(\tau) \, d\tau = m(t), \quad t \in (0, T)$$
with given $m$. It is well-known that in case $m \in L^p(0, T)$ with $p > 1$ the solution $k$
exists, is unique and belongs to $L^p(0, T)$ (see e.g. [5]).

It is possible to transform the direct problem (1), (5) to a form that contains the kernel $k$ instead of $m$. Indeed, observing the formula $(I + k)(I - m) = I$, where $I$ is the unity operator we get

**Lemma 1.** In case all $t$-dependent functions and their derivatives involved in equations (1) and (5) belong to the space $L^p(0, T)$ with respect to $t$ for any $x$ with some $p > 1$, these equations are equivalent to

(13) $\beta (u_0 + k^* u_t) = Au_0 + f$ in $Q$, $u_0 = u_0$ in $\Omega \times \{0\}$, $B_1 u = g$ in $S$,

where

(14) $f = \chi + k^* \chi$,

(15) $B_1 = B$, $g = b$ in case I,

(16) $B_1 u = \omega \cdot \nabla x u$, $g = b + k^* b$ in case II.

Now we are going to prove certain basic results for the direct problem of the form

(13). For the sake of generality, we allow the kernel $k$ and the coefficients $\beta, a_{ij}$ and $a_i$ to depend on both $x$ and $t$ in Theorems 1 and 2 below.

**Theorem 1.** Assume (3) and (4) in $Q$. Then the following assertions are valid.

(i) Let $\beta, a_{ij}, a_i, a \in \mathcal{C}(\overline{\Omega})$, $k \in L^1(0, T; L^{\infty} (\Omega))$, $f \in L^p(Q)$, $u_0 \in W^{2-\frac{p}{2}}_p (\Omega)$ and

$g \in W^{2-\frac{p}{2} - \nu, 1 - \frac{p}{2}}_p (S)$ with some $p \in (1, \infty)$. Moreover, in case I $p \neq 4$ and the consistency condition $u_0 = g$ hold in $\Gamma \times \{0\}$ if $p > \frac{4}{\nu}$ and in case II $p \neq 3$ and the consistency condition $\omega \cdot \nabla u_0 = g$ hold in $\Gamma \times \{0\}$ if $p > 3$. Then the problem (13) has a unique solution in the space $W^{2-\frac{p}{2}}_p (Q)$.

This solution satisfies the estimate

(17) $\|u\|_{W^{2-\frac{p}{2}}_p (Q)} \leq C_1 \left( \|f\|_{L^p (Q)} + \|u_0\|_{W^{2-\frac{p}{2}}_p (\Omega)} + \|g\|_{W^{2-\frac{p}{2} - \nu, 1 - \frac{p}{2}}_p (S)} \right),$

where $C_1$ is a constant depending on $\beta, a_{ij}, a_i, a$ and $k$.

(ii) Let $\beta, a_{ij}, a_i, a \in C^1(\overline{\Omega})$ and $k \in L^\infty(0, T; C^4 (\Omega))$ with some $l \in (0, 1)$.

Moreover, let $f \in C^4 (Q)$, $u_0 \in C^{2+1,l} (\Omega)$, $g \in C^{2+1,l - \nu, 1 - \frac{p}{2}}_p (S)$ and in case I the consistency conditions $u_0 = g$, $\beta g = Au_0 + f$ hold in $\Gamma \times \{0\}$ in case II the consistency condition $\omega \cdot \nabla u_0 = g$ hold in $\Gamma \times \{0\}$. Then the solution of (13) belongs to $C^{2+1,l + \frac{p}{2}} (Q)$ and satisfies the estimate

(18) $\|u\|_{C^{2+1,l + \frac{p}{2}} (Q)} \leq C_2 \left( \|f\|_{C^4 (Q)} + \|u_0\|_{C^{2+1,l} (\Omega)} + \|g\|_{C^{2+1,l - \nu, 1 - \frac{p}{2}}_p (S)} \right)$

with some constant $C_2$ depending on $\beta, a_{ij}, a_i, a$ and $k$.

**Proof.** Let us denote $Q_t = \Omega \times (0, t)$. To prove (i), we need the relation

(19) $\|k^* v\|_{L^p (Q_t)} \leq \int_0^t \|k(\cdot, t - \tau)\|_{L^\infty (\Omega)} \|v\|_{L^p (Q_\tau)} d\tau$, $t \in (0, T)$, $v \in L^p (Q_t)$.

Denoting $\tilde{k}(t) = k(\cdot, t)$, $\tilde{v}(t) = v(\cdot, t)$, $\|\tilde{k}(t)\|_{L^\infty (\Omega)} = \|\tilde{v}(t)\|_{L^p (\Omega)}$, using the following property of the Bochner integral: $\|\int_0^t w(x, \tau) d\tau\|_{L^p (\Omega)} \leq$
\[ \int_0^t \| w(s, \tau) \|_{L^p(\Omega)} \, d\tau \] for functions \( w(s, \cdot) \in L^1(0, T; L^p(\Omega)) \) and the H\"older's inequality, the sought relation (19) can be deduced by means of the following computations:

\[
\| k \ast v \|_{L^p(\Omega)} = \left[ \int_0^t \right. \left( \int_0^\tau \| \tilde{k}(s) \|_{L^p(\Omega)} \, ds \right) \, d\tau \right]^{\frac{1}{p}} \leq I,
\]

where

\[
I = \left[ \int_0^t \right. \left( \int_0^\tau \| \tilde{k}(s) \|_{L^p(\Omega)} \, ds \right) \, d\tau \right]^{\frac{1}{p}} \leq \left[ \int_0^t \| \tilde{k}(s) \|_{L^p(\Omega)} \, ds \right] \left[ \int_0^\tau \| \tilde{k}(s) \|_{L^p(\Omega)} \, ds \right] \leq \left[ \int_0^t \| \tilde{k}(s) \|_{L^p(\Omega)} \, ds \right] \left[ \int_0^\tau \| \tilde{k}(s) \|_{L^p(\Omega)} \, ds \right].
\]

By Theorem 5.4 in [14], under the assumptions of (i) problem (13) in case \( k \neq 0 \) has a unique solution \( \hat{u} \) in the space \( W^{2,1}_p(\Omega) \). Thus, (13) for \( u \in W^{2,1}_p(\Omega) \) in case \( k \neq 0 \) is equivalent to the following problem for the difference \( v = u - \hat{u} \in W^{2,1}_p(\Omega) \):

(20) \hspace{1cm} \beta u_t = Av - \beta (k \ast (v + \hat{u})) \in Q, \hspace{0.5cm} v = 0 \text{ in } \Omega \times \{0\}, \hspace{0.5cm} B_1 v = 0 \text{ in } S.

Let \( F \) stand for the operator that assigns to a function \( f \) the solution of the problem (13) in the case \( k = 0, u_0 = 0, g = 0 \). By Theorem 5.4 in [14], \( F \in \mathcal{L}(L^p(\Omega), W^{2,1}_p(\Omega)) \). This implies that the problem (20) in \( W^{2,1}_p(\Omega) \) is equivalent to the following fixed-point equation:

(21) \hspace{1cm} v = Fv + F\hat{u}, \hspace{1cm} \text{where } Fv = -F(\beta (k \ast u_t)) \hspace{1cm} \text{and} \hspace{1cm} F \in \mathcal{L}(W^{2,1}_p(\Omega)).

and \( F \in \mathcal{L}(W^{2,1}_p(\Omega)) \). Let \( r \in (0, T] \) and define the cutting operator \( P_r \) by the formula \( P_r v = \begin{cases} v & \text{in } Q_r \setminus \{0\}, \\ 0 & \text{in } Q \setminus Q_r. \end{cases} \). Observing that \( Fv = FP_r v \) in \( Q_r \) and using (19) we deduce the estimate

\[
\| Fv \|_{W^{2,1}_p(\Omega)} = \| F(\beta (k \ast u_t)) \|_{W^{2,1}_p(\Omega)} = \| FP_r (\beta (k \ast u_t)) \|_{W^{2,1}_p(\Omega)} \leq \| FP_r (\beta (k \ast u_t)) \|_{L^\infty(\Omega)} \leq \| F \| \| \beta (k \ast u_t) \|_{L^\infty(\Omega)} \leq C_2 \int_0^T \| \tilde{k}(t - \tau) \|_{L^\infty(\Omega)} \| v \|_{W^{2,1}_p(\Omega)} \, d\tau
\]

with \( C_2 = \| F \| \| \beta \|_{C(\tilde{Q})} \) and \( \tilde{k}(t) = k(\cdot, t) \), as before. Now we define the weighted norms \( \| v \|_\sigma = \sup_{0 < t < T} e^{-\sigma t} \| v \|_{W^{2,1}_p(\Omega)} \), \( \sigma > 0 \), in the space \( W^{2,1}_p(\Omega) \) and deduce the estimate

(22) \hspace{1cm} \| Fv \|_\sigma \leq C_2 \sup_{0 < t < T} e^{-\sigma t} \int_0^T \| \tilde{k}(t - \tau) \|_{L^\infty(\Omega)} \| v \|_{W^{2,1}_p(\Omega)} \, d\tau
\]

\[
= C_2 \sup_{0 < t < T} \int_0^T e^{-\sigma(t - \tau)} \| \tilde{k}(t - \tau) \|_{L^\infty(\Omega)} e^{-\sigma \tau} \| v \|_{W^{2,1}_p(\Omega)} \, d\tau \leq C_\sigma \| v \|_\sigma
\]

where \( C_\sigma = C_2 \int_0^T e^{-\sigma \tau} \| \tilde{k}(\tau) \|_{L^\infty(\Omega)} \, d\tau. \) Since \( \tilde{k} \in L^1(0, T; L^\infty(\Omega)) \), there exists \( \sigma_0 > 0 \) such that \( C_\sigma < \frac{1}{2} \). Consequently, \( F \) is a contraction and the equation (21), which is equivalent to (13), has a unique solution in \( W^{2,1}_p(\Omega) \). Moreover, from (21) and (22) we deduce \( \| v \|_{\sigma_0} \leq \| Fv \|_{\sigma_0} + \| F\hat{u} \|_{\sigma_0} \leq \frac{1}{2} (\| v \|_{\sigma_0} + \| \hat{u} \|_{\sigma_0}). \) This implies \( \| v \|_{\sigma_0} \leq \| \hat{u} \|_{\sigma_0}. \) Since \( \| \hat{u} \|_{W^{2,1}_p(\Omega)} \) is bounded by the right-hand side of (17)
(see Theorem 5.4 in [14]) and \( \| \cdot \|_\sigma \) is equivalent to the usual norm in \( W^2_\sigma (Q) \), we obtain the estimate (17).

To prove (ii), we need the relation

\[
\| k \ast v \|_{C^{\alpha, \beta}(Q_t)} \leq C_3 \left[ \int_0^T \left\{ \| k(\cdot, t - \tau) \|_{C^{\alpha}(\Omega)} \| v \|_{C^{\beta, \beta}(Q_t)} \right\} \frac{dr}{\tau^{\alpha_+}} \right]^{\frac{\alpha}{\alpha_+}},
\]

\( t \in (0, T), \; v \in C^{\beta, \beta}(Q) \)

that holds with some constant \( C_3 \). It follows by splitting \( \| k \ast v \|_{C^{\alpha, \beta}(Q_t)} = N_1 + N_2 \), where

\[
N_1 = \sup_{(x, s) \in Q_t} \| (k \ast v)(x, s) \| + \sup_{(x_1, s) \in Q_t} \| (k \ast v)(x_1, x_2; s) \|
\]

\[
\leq \sup_{s \in (0, t)} \int_0^s \sup_{x \in \Omega} \| k(x, s - \tau) \| \sup_{(x, \eta) \in Q_\eta} \| v(t, x_1, x_2; \eta) \| \frac{dr}{\tau^{\alpha_+}} \]

\[
+ \sum_{s \in (0, t)} \int_0^s \sup_{x \in \Omega} \| k(x, s - \tau) \| \sup_{(x_1, \eta_1), (x_2, \eta_2) \in Q_\eta} \| v(t, x_1, \eta_1; \eta_2) \| \frac{dr}{\tau^{\alpha_+}} \]

\[
\leq \int_0^T \| k(\cdot, t - \tau) \|_{C^{\alpha}(\Omega)} \| v \|_{C^{\beta, \beta}(Q_t)} \frac{dr}{\tau^{\alpha_+}}
\]

\[
\leq C_4 \left[ \int_0^T \left\{ \| k(\cdot, t - \tau) \|_{C^{\alpha}(\Omega)} \| v \|_{C^{\beta, \beta}(Q_t)} \right\} \frac{dr}{\tau^{\alpha_+}} \right]^{\frac{\alpha}{\alpha_+}},
\]

with some constant \( C_4 \). In these estimations we used the Hölder inequality and the fact that the convolution of a nonnegative and a nonnegative nondecreasing function is nondecreasing.

By Theorem 4.9 in [14] under the assumptions (ii) we have \( \hat{u} \in C^{\alpha+2, \beta+1}(Q), \; F \in \mathcal{L}(C^{\alpha, \beta}(Q), C^{\alpha+2, \beta+1}(Q)), \; F \in \mathcal{L}(C^{\alpha+2, \beta+1}(Q)) \) and the problem (13) for \( u \in C^{\alpha+2, \beta+1}(Q) \) in case \( k \neq 0 \) is equivalent to the fixed-point equation (21) for \( v = u - \hat{u} \in C^{\alpha+2, \beta+1}(Q) \). Let \( t \in (0, T) \) and define the operator \( \tilde{P} \) such that \( \tilde{P}v = v \) in \( Q_t \) and \( \tilde{P}v(x, \tau) = v(x, t) \) for \( \tau \in [t, T] \). Then,

\[
\| \tilde{P}v \|_{C^{\alpha, \beta}(Q_t)} = \| v \|_{C^{\alpha, \beta}(Q_t)}
\]

and \( Fv = F\tilde{P}v \) in \( Q_t \). Thus, using (23) we obtain

\[
\| Fv \|_{C^{\alpha+2, \beta+1}(Q_t)} = \| F\tilde{P}v(\beta(\cdot, v)) \|_{C^{\alpha+3, \beta+1}(Q_t)} \leq \| F\tilde{P}v(\beta(\cdot, v)) \|_{C^{\alpha+3, \beta+1}(Q_t)}
\]

\[
\| Fv \|_{C^{\alpha+2, \beta+1}(Q_t)} \leq \| F\tilde{P}v(\beta(\cdot, v)) \|_{C^{\alpha+3, \beta+1}(Q_t)}
\]

\[
\| Fv \|_{C^{\alpha+2, \beta+1}(Q_t)} \leq \| F\tilde{P}v(\beta(\cdot, v)) \|_{C^{\alpha+3, \beta+1}(Q_t)}
\]
\[ \|F\|_{C^{i+\frac{1}{2}}(Q_i)} = \|F\|_{C^{i+\frac{1}{2}}(Q)} \leq C_5 \left[ \int_0^T \left\{ \|k(\cdot, t - \tau)\|_{C^i(\Omega)} \|v\|_{C^{i+\frac{1}{2}}(Q_i)} \right\} \frac{d\tau}{1 + \tau^2} \right]^{\frac{1}{2}} \]

Using the weighted norms \( \|w\|_\sigma = \sup_{0 < t < T} e^{-\sigma t} \|w\|_{C^{i+\frac{1}{2}}(Q_i)} \), \( \sigma > 0 \), in the space \( C^{i+\frac{1}{2}}(Q) \), we deduce

\[ \|v\|_{\sigma} \leq C_5 \sup_{0 < t < T} \left[ \int_0^T \left\{ e^{-\sigma(t-\tau)} \|k(\cdot, t - \tau)\|_{C^i(\Omega)} e^{-\sigma \tau} \|v\|_{C^{i+\frac{1}{2}}(Q_i)} \right\} \frac{d\tau}{1 + \tau^2} \right]^{\frac{1}{2}} \]

Since \( k \in L^2(0, T; C^i(\Omega)) \), the coefficient \( \int_0^T \left\{ e^{-\sigma(t-\tau)} \|k(\cdot, t - \tau)\|_{C^i(\Omega)} \right\} \frac{d\tau}{1 + \tau^2} \) is small for large \( \sigma \). Owing to this, the proof can be finished as in case (i) making use of the fixed-point argument and an estimate for \( \|v\|_{C^{i+\frac{1}{2}}(Q)} \) in Theorem 4.9 of [14].

**Theorem 2.** Assume (3), (4), \( \beta, a_i, a, \alpha \in C(\overline{Q}) \) and

\[ k \in W_p^l(0, T; L^\infty(\Omega)), \quad k \geq 0, \quad k_t \leq 0. \]

Let \( u \in W_p^{2,1}(Q) \) with some \( p \in (1, \infty) \) solve the problem (13) and \( u_0 \geq 0, \quad g \geq 0, \quad f \geq 0. \) Then the following assertions are valid:

(i) \( u \geq 0; \)

(ii) if, in addition, \( \beta, a_i, a, \alpha \in C^l(\overline{Q}) \) with some \( l \in (0, 1) \) and either \( f \neq 0 \) or \( g \neq 0 \), then \( u(\cdot, T) > 0 \) in \( \Omega \) in case I and \( u(\cdot, T) > 0 \) in \( \overline{\Omega} \) in case II.

**Proof.** It consists of 4 steps.

1. step: proof the assertion (i) in case \( u \in C^{2,1}(\overline{Q}), \quad a \leq 0 \) and \( k \) satisfies the conditions

\[ k \in W_p^l(0, T; C(\overline{\Omega})), \quad k \geq 0, \quad k_t \leq 0. \]

Since \( u = u_0 \geq 0 \) in \( \Omega \times (0, T) \), there exists

\[ t_0 = \sup\{t : u(x, \tau) \geq 0 \text{ for } (x, \tau) \in \Omega \times [0, t], \ 0 \leq t \leq T\}. \]

In case the assertion \( u \geq 0 \) holds in \( Q \), we have \( t_0 = T. \) Suppose on the contrary that \( t_0 < T. \) Then, we fix some \( h \in (0, T - t_0) \) and define the sets \( V_{t_0, h} = \Omega \times (t_0, t_0 + h) \) and \( \overline{V}_{t_0, h} = \overline{\Omega} \times \{t_0, t_0 + h\}. \) By this definition, there exists \( (x_{t_0}^h, t_{t_0}^h) \in V_{t_0, h} \) such that \( u(x_{t_0}^h, t_{t_0}^h) < 0. \) Let

\[ v(\cdot, t) = u(\cdot, t) + \mu_h(t - t_0 - h), \quad \mu_h = \frac{u(x_{t_0}^h, t_{t_0}^h)}{2h} > 0. \]

Then we have

\[ u(\cdot, t) - \mu_h h \leq v(\cdot, t) \leq u(\cdot, t) \text{ for } (x, t) \in \overline{V}_{t_0, h}. \]

Observing (27) and the inequality \( u(x_{t_0}^h, t_{t_0}^h) < 0 \) we see that for all \( (x, t) \in \overline{V}_{t_0, h} \) such that \( u(x, t) \geq 0. \) the relations

\[ v(x, t) \geq u(x, t) - \mu_h h \geq -\mu_h h = \frac{u(x_{t_0}^h, t_{t_0}^h)}{2} > u(x_{t_0}^h, t_{t_0}^h) \geq v(x_{t_0}^h, t_{t_0}^h) \]

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are valid. They imply that

$$\text{function } v \text{ cannot attain its minimum over } V_{t_0, h}$$

in a point \((x, t)\) where \(u(x, t) \geq 0\).

In particular, (28) implies that \(v\) cannot attain its minimum over \(V_{t_0, h}\) on the subset \(V_{t_0, h} \setminus V_{t_0, h} = \Omega \times \{t_0\}\), because there \(u \geq 0\) in view of the definition of \(t_0\). Therefore,

$$\exists (x_h, t_h) \in V_{t_0, h} : v(x_h, t_h) \leq v(x, t) \text{ for all } (x, t) \in V_{t_0, h}.$$  

Moreover, \(v(x_h, t_h) < 0\), because \(v(x_h, t_h) \leq v(x^*_0, t^*_0) \leq u(x^*_0, t^*_0)\) and \(u(x^*_0, t^*_0) < 0\).

Let us show that \(x = x_h\) is the stationary minimum point of \(v(x, t_h)\), i.e.

$$\nabla_x v(x_h, t_h) = 0.$$  

This relation may fail only in case the minimum occurs in the lateral boundary of \(V_{t_0, h}\), i.e. when \(x_h \in \Gamma\). In case I we have \(u = g \geq 0\) for \(x \in \Gamma\) and, by statement (28), \(x_h\) cannot belong to \(\Gamma\). Thus, it remains to show (29) for the case II when \(x_h \in \Gamma\). In this case \(\omega \cdot \nabla_v v = \omega \cdot \nabla_x u = g\) in \(\Gamma\). Note that then the inequality \(g(x_h, t_h) > 0\) cannot hold, because otherwise \(v\) is strictly decreasing in the inner direction \(-\omega(x_h)\) at \((x_h, t_h)\) which implies that \(v(x_h, t_h)\) cannot be the minimum of \(v\). Consequently, due to the assumption \(g \geq 0\), it holds \(g(x_h, t_h) = 0\) and we have \(\omega(x_h) \cdot \nabla_x v(x_h, t_h) = 0\). In addition, we also have \(\tau \cdot \nabla_x v(x_h, t_h) = 0\), where \(\tau\) is an arbitrary tangential direction of \(\Gamma\) at \(x_h\), because \(x = x_h\) is the minimum point of the \(x\)-dependent function \(v(x, t_h)\) over the set \(\Gamma\). These relations yield \(\xi \cdot \nabla_x v(x, t_h) = 0\), where \(\xi\) is any direction, hence (29).

Now we are going to estimate the operator \(Lu = \beta(u_t + k * u_t) - Au\) of the equation (13) termwise at \((x, t) = (x_h, t_h)\). By (26) we have \(u_t(x_h, t_h) = v_t(x_h, t_h) - \mu_h\). Since \(t_h\) is the minimum point of the \(t\)-dependent function \(v(x, t)\) in the half-interval \([t_0, t_0 + h]\), it holds \(u_t(x_h, t_h) \leq 0\). Thus, we obtain

$$\mu_h \leq u_t(x_h, t_h).$$  

By (2) and (26) we have \(Au = \sum_{i,j=1}^{N} a_{ij} u_{x_i, x_j} + \sum_{j=1}^{N} a_j v_{x_j} + a(v - \mu_h (t - t_0 - h))\). Since \(x = x_h\) is the stationary minimum point of \(v(x, t_h)\) and the principal part of \(A\) is elliptic (see (3)), it holds \(\sum_{i,j=1}^{N} a_{ij}(x_h, t_h) u_{x_i, x_j}(x_h, t_h) \geq 0\) and \(\sum_{j=1}^{N} a_j(x_h, t_h) v_{x_j}(x_h, t_h) = 0\). Moreover, in view of \(a \leq 0\) and \(v(x_h, t_h) < 0\) we have \(a(x_h, t_h) v(x_h, t_h) \geq 0\). By means of these relations and \(t_h \in (t_0, t_0 + h]\) we deduce the estimate

$$-Au(x_h, t_h) \leq a(x_h, t_h) \mu_h (t_h - t_0 - h) \leq C^0 \mu_h$$

where \(C_0 = \|a\|_{C^0(\Omega)}\).

It remains to deal with the term \((k * u_t)(x_h, t_h)\) in \(Lu(x_h, t_h)\). Integrating by parts we have

$$\int_0^{t_h} k(x_h, t_h - \tau) u_t(x_h, \tau) d\tau = k(x_h, 0) u(x_h, t_h) - k(x_h, t_h) u_0(x_h)$$

$$+ \int_0^{t_h} k_t(x_h, t_h - \tau) u(x_h, \tau) d\tau + \int_0^{t_h} k_t(x_h, t_h - \tau) u(x_h, \tau) d\tau.$$  

In this relation \(-k(x_h, t_h) u_0(x_h) \leq 0\) and \(\int_0^{t_h} k_t(x_h, t_h - \tau) u(x_h, \tau) d\tau \leq 0\), because \(k \geq 0\), \(k_t \leq 0\), \(u_0 \geq 0\) and \(u(x_h, \tau) \geq 0\) for \(\tau \in [0, t_0]\) by the definition of \(t_0\).
Consequently,
\[ \int_0^{t_h} k(x_h, t_h - \tau) u_\tau(x_h, \tau) d\tau \leq k(x_h, 0) u(x_h, t_h) + \int_0^{t_h} k_t(x_h, t_h - \tau) u(x_h, \tau) d\tau. \]
Substituting here \( u(x_h, t) \) by \( v(x_h, t) - \mu_h(t - t_0 - h) \) (see (26)), we have
\[ (32) \int_0^{t_h} k(x_h, t_h - \tau) u_\tau(x_h, \tau) d\tau \]
\[ \leq k(x_h, 0) v(x_h, t_h) + \int_0^{t_h} k_t(x_h, t_h - \tau) v(x_h, \tau) d\tau - \]
\[ -\mu_h \left[ k(x_h, 0)(t_h - t_0 - h) + \int_0^{t_h} k_t(x_h, t_h - \tau)(\tau - t_0 - h) d\tau \right]. \]
In this estimate we analyze separately the term \( \int_0^{t_h} k_t(x_h, t_h - \tau)v(x_h, \tau)d\tau. \) To this end, define the following subsets of \([t_0, t_h]\):
\[ U_+^h = \{ \tau : \tau \in [t_0, t_h], \ v(x_h, \tau) \geq 0 \}, \quad U_-^h = \{ \tau : \tau \in [t_0, t_h], \ v(x_h, \tau) < 0 \}. \]
Taking account of \( k_t \leq 0 \) and the fact that \( v(x_h, t_h) < 0 \) is the minimum of \( v(x_h, t) \) on the interval \([t_0, t_h]\), we deduce
\[ \int_0^{t_h} k_t(x_h, t_h - \tau)v(x_h, \tau)d\tau \]
\[ = \int_{U_+^h} k_t(x_h, t_h - \tau)v(x_h, \tau)d\tau + \int_{U_-^h} k_t(x_h, t_h - \tau)v(x_h, \tau)d\tau \]
\[ \leq \int_{U_-^h} k_t(x_h, t_h - \tau)v(x_h, \tau)d\tau \leq \int_0^{t_h} k_t(x_h, t_h - \tau)d\tau \cdot v(x_h, t_h) \]
\[ = (k(x_h, t_h - t_0) - k(x_h, 0)) v(x_h, t_h). \]
Thus, due to \( k \geq 0 \) we get
\[ k(x_h, 0) v(x_h, t_h) + \int_0^{t_h} k_t(x_h, t_h - \tau)v(x_h, \tau)d\tau \leq k(x_h, t_h - t_0) v(x_h, t_h) \leq 0 \]
and from (32) we finally obtain
\[ (33) \int_0^{t_h} k(x_h, t_h - \tau) u_\tau(x_h, \tau) d\tau \]
\[ \leq -\mu_h \left[ k(x_h, 0)(t_h - t_0 - h) + \int_0^{t_h} k_t(x_h, t_h - \tau)(\tau - t_0 - h) d\tau \right]. \]
Let’s return to the operator \( L \). Making use of (30), (31), (33), the assumption (4) and the relation \( 0 < t_h - t_0 \leq h \) we obtain
\[ L u(x_h, t_h) \leq \mu_h \left\{ \begin{array}{l} -\beta(x_h) \\
-\beta(x_h) \left[ k(x_h, 0)(t_h - t_0 - h) + \int_{t_0}^{t_h} k_t(x_h, t_h - \tau)(\tau - t_0 - h) d\tau \right] + C \end{array} \right\} \]
\[ \leq \mu_h \left\{ \begin{array}{l} -\beta_0 + C \gamma h \left( k(x_h, 0) + \| k_t \|_{L^1([0,T]; C(M))} + C \right) \right\}. \]
In case \( h > 0 \) is sufficiently small, due to the inequalities \( \mu_h > 0 \) and \( \beta_0 > 0 \) the relation \( L u(x_h, t_h) < 0 \) holds. But this contradicts to the assumption \( L u = f \geq 0 \).
Consequently, the supposition \( t_0 < T \) was not right. We have \( t_0 = T \), which by the definition of \( t_0 \) implies \( u \geq 0 \).

2. step: proof of the assertion (i) in case \( u \in C^{2,1}(\Omega) \), \( u_0 = 0 \) and \( k \) satisfies (25).

Let us define \( \tilde{u} = e^{-\sigma t}u \), \( \tilde{k}(x, t) = e^{-\sigma t}k(x, t) + \sigma \int_0^t e^{-\sigma \eta}k(x, \eta)d\eta \), \( \tilde{f} = e^{-\sigma t}f \), \( \tilde{g} = e^{-\sigma t}g \), where \( \sigma = \frac{\|k\|_{L^\infty(\Omega)}}{S_0} \). Then \( \tilde{u} \) solves the problem

\[
\beta(\tilde{u}_t + \tilde{k} \ast \tilde{u}_x) = \tilde{A}\tilde{u} + \tilde{f} \quad \text{in} \ Q, \quad \tilde{u} = u_0 \quad \text{in} \ \Omega \times \{0\}, \quad B_1\tilde{u} = \tilde{g} \quad \text{in} \ S,
\]

where \( \tilde{A}\tilde{u} = \sum_{i,j=1}^n a_{ij}\tilde{u}_{x_i x_j} + \sum_{j=1}^n a_j \tilde{u}_{x_j} + \tilde{a}\tilde{u} \) and

\[
\tilde{a}(x, t) = a(x, t) - \sigma \beta(x, t) \left[ 1 + \int_0^t e^{-\sigma \eta}k(x, \eta)d\eta \right] \leq 0.
\]

The kernel \( \tilde{k} \) satisfies (25) and \( \tilde{g} \geq 0, \tilde{f} \geq 0 \). Using part 1 of the proof, we get \( \tilde{u} \geq 0 \). This implies \( u \geq 0 \).

3. step: proof of assertion (ii) in the general case.

It is enough to prove this assertion for \( p \in \left(1, \frac{2}{3}\right) \). Let us consider the problems

\[
\beta u_t = (A - k(\cdot, 0)\beta) u \quad \text{in} \ Q, \quad u = u_0 \quad \text{in} \ \Omega \times \{0\}, \quad B_1u = g \quad \text{in} \ S, \quad (34)
\]

\[
\beta(\tilde{u}_t + k \ast \tilde{u}_x) = \tilde{A}\tilde{u} + f \quad \text{in} \ Q, \quad \tilde{u} = 0 \quad \text{in} \ \Omega \times \{0\}, \quad B_1\tilde{u} = 0 \quad \text{in} \ S, \quad (35)
\]

where \( f_\alpha = f - \beta(k \ast u) + \beta k(\cdot, 0)u_0 \). Since \( u \in W^{2,1}_p(\Omega) \), by embedding theorems [10] we have \( u_0 = u|_{\Gamma \times \{0\}} \in W^{2,1}_p(\Omega) \) and \( g = B_1u|_S \in W^{2,1}_p(\Gamma \times \{0\}) \).

Thus, by Theorem 1 (i), problem (34) has a unique solution \( \tilde{u} \in W^{2,1}_p(\Omega) \). Further, due to the assumptions of the theorem and \( \tilde{u} \in W^{2,1}_p(\Omega) \), it holds \( f_\alpha \in L^p(\Omega) \) and by Theorem 1 (i), again, problem (35) has a unique solution \( \tilde{u} \in W^{2,1}_p(\Omega) \). Adding (34) and (35) and integrating by parts in the convolution term in \( f_\alpha \), we see that \( u + \tilde{u} \) solves (13), hence by the uniqueness, \( u = \tilde{u} \).

By the well-known extremum principle for weak solutions of parabolic equations (e.g. [10] Ch. 3, Theorem 7.2) we obtain \( \tilde{u} \geq 0 \). This together with the assumptions of theorem implies \( f_\alpha \geq 0 \).

Further, let us choose some functions \( \beta^\alpha, a^\alpha, a_j^\alpha, a_m^\alpha, \hat{f}^\alpha, \hat{k}^\alpha \in C^\infty(\overline{\Omega}) \) such that

\[
\|\beta^\alpha - \beta\|_{C(\overline{\Omega})}, \|a^\alpha - a\|_{C(\overline{\Omega})}, \|a_j^\alpha - a_j\|_{C(\overline{\Omega})}, \|a_m^\alpha - a_m\|_{C(\overline{\Omega})}, \|f^\alpha - f\|_{L^p(\Omega)}, \|k^\alpha - k\|_{W^{2,1}_p(\Gamma \times \{0\})} \to 0 \quad \text{as} \quad m \to \infty
\]

and \( \hat{f}^\alpha = 0 \) in \( \Gamma \times \{0\} \). Then, by virtue of (3) and (4), for sufficiently large \( m \) there hold the relations

\[
\sum_{i,j=1}^n a_{ij}^\alpha \lambda_i \lambda_j \geq \frac{\xi}{2} |\lambda|^2 \quad \text{for any} \quad \lambda \in \mathbb{R}^n \quad \text{and} \quad \beta^\alpha \geq \frac{\beta_0}{2} \quad \text{in} \ \overline{\Omega}.
\]

In addition, we define \( f^\alpha(x, t) = \begin{cases} \hat{f}^\alpha(x, t) & \text{if} \ \hat{f}^\alpha(x, t) \geq 0 \\ 0 & \text{if} \ \hat{f}^\alpha(x, t) < 0 \end{cases} \). Then \( f^\alpha \in C^{1,\frac{1}{2}}(\Omega) \) for any \( l \in \{0, 1\} \) and \( f^\alpha = 0 \) in \( \Gamma \times \{0\} \). Moreover, due to the inequality \( f_\alpha \geq 0 \) it holds \( |f^\alpha - f_\alpha| \leq |\hat{f}^\alpha - f_\alpha| \) in \( \Omega \). Therefore, using (36) we obtain

\[
\|f^\alpha - f_\alpha\|_{L^p(\Omega)} \leq |\hat{f}^\alpha - f_\alpha|_{L^p(\Omega)} \to 0 \quad \text{as} \quad m \to \infty.
\]
Finally, we define \( k^m(x, t) = \int_T^t \kappa^m(x, \tau)d\tau + q^m(x) \) with
\[
\kappa^m(x, t) = \begin{cases} 
\hat{k}^m(x, t) & \text{if } \hat{k}^m(x, t) \leq 0 \\
0 & \text{if } \hat{k}^m(x, t) > 0 
\end{cases}
, \quad q^m(x) = \begin{cases} 
\hat{q}^m(x, T) & \text{if } \hat{k}^m(x, T) \geq 0 \\
0 & \text{if } \hat{k}^m(x, T) < 0 
\end{cases}
.
\]
Then \( k^m \in W^1_t(0, T; C^0(\Omega)) \) for all \( t \in (0, 1) \). Moreover, since \( k \geq 0 \) and \( k_t \leq 0 \), we obtain \( |k^m - k| \leq |\hat{k}^m - \hat{k}| \) in \( Q \) and \( |q^m_k(T) - k(T)| \leq |\hat{q}^m_k(T) - \hat{k}(T)| \) in \( \Omega \).

Observing these relations and (36) we deduce
\[
(38) \quad \|k^m - k\|_{L^1(0, T; L^\infty(\Omega))} \leq \int_0^T \operatorname{ess sup}_{x \in \Omega} \left[ \int_t^T |\kappa^m(x, \tau) - k_r(x, \tau)|d\tau + |q^m(x) - k(x, T)| \right] dt \leq T \left[ \|\hat{k}^m_k - \hat{k}\|_{L^1(0, T; L^\infty(\Omega))} + \|\hat{q}^m_m(T) - k(T)\|_{L^\infty(\Omega)} \right] \to 0 \quad \text{as} \quad m \to \infty.
\]

Now let us consider the following approximating problems of (35):
\[
\beta^m(u^m_k + k \ast u^m) = A^m u^m + f^m \quad \text{in} \quad Q, \\
u^m = 0 \quad \text{in} \quad \Omega \times \{0\}, \quad B_1 u^m = 0 \quad \text{in} \quad S,
\]
where \( A^m v = \sum_{i,j=1}^n a^m_{ij} v_{x_i x_j} + \sum_{j=1}^n a^m_{i} v_{x_j} + a^m v \). Observing the proved regularity of the data of these problems and Theorem 1 (ii) we conclude that for each integer \( m \), (39) has the unique solution \( u^m \in C^{2+1,1+\frac{\epsilon}{2}}(Q) \). Subtracting (35) from (39) we obtain the following problems for the differences \( v^m = u^m - \bar{u} \):
\[
\beta(v^m + k \ast v^m) = A v^m + \phi^m \quad \text{in} \quad Q, \quad v^m = 0 \quad \text{in} \quad \Omega \times \{0\}, \quad B_1 v^m = 0 \quad \text{in} \quad S,
\]
where
\[
\phi^m = (A^m - A)(v^m + \bar{u}) - (\beta^m - \beta)v^m + \bar{u} + k \ast (v^m + \bar{u}) \\
- \beta^m - \beta)((k^m - k) \ast (v^m + \bar{u})) + f^m - f_0.
\]

Using Theorem 1 (i) for these problems we deduce
\[
\|v^m\|_{W^{2,1}_r(Q)} \leq C_1 \|\phi^m\|_{L^r(Q)} \leq C_1 \|f^m - f_0\|_{L^r(Q)} + C_1 \left[ \|a^m_{ij} - a_{ij}\|_{C^0(\Omega)} + \|a^m - a\|_{C^0(\Omega)} + \|\beta^m - \beta\|_{C^0(\Omega)} \right] + (1 + \|\beta^m - \beta\|_{C^0(\Omega)}) \|k^m - k\|_{L^1(0, T; L^\infty(\Omega))} \leq \|v^m\|_{W^{2,1}_r(Q)} + \|\bar{u}\|_{W^{2,1}_r(Q)}
\]
with some constant \( C_1 \). Taking here (36) - (38) into account, we get
\[
(40) \quad \|v^m\|_{W^{2,1}_r(Q)} = \|u^m - \bar{u}\|_{W^{2,1}_r(Q)} \to 0 \quad \text{as} \quad m \to \infty.
\]
Note that \( f^m \geq 0, \quad k^m \geq 0 \) and \( k^m_t \leq 0 \), by the definitions. Thus, since \( u^m = 0 \) on \( \Omega \times \{0\} \), using step 2 of the proof for the solution of the problem (39), we get \( u^m \geq 0 \). This with (40) implies \( \bar{u} \geq 0 \). Since \( \bar{u} \geq 0 \), we obtain \( u = \bar{u} + \bar{u} \geq 0 \). The assertion (i) is proved.

4. step: proof of assertion (ii).

Firstly, let us consider the case \( f \neq 0 \). Then, due to (11) and (10), there exists an open ball \( U \subseteq Q \) and \( \epsilon > 0 \) such that \( f \geq \epsilon \) in \( U \), hence we can choose some
$f^1 \in C^{\infty}(\overline{Q})$ so that $f^1 = 0$ in $Q \setminus U$ and $0 < f^1 \leq \frac{1}{x}$ in $U$. Further, let us define $c_k = \text{ess sup} \, k(x, 0)$ and consider the following problems:

(41) \hspace{1cm} \beta u^1_t = (A - c_k \beta) u^1 + f^1 \text{ in } Q, \quad u^1 = 0 \text{ in } \Omega \times \{0\}, \quad B_1 u^1 = 0 \text{ in } S,

(42) \hspace{1cm} \beta(\overline{u} + k \ast \overline{u}) = A \overline{u} + \overline{f}_{xu} \text{ in } Q, \quad \overline{u} = u_0 \text{ in } \Omega \times \{0\}, \quad B_1 \overline{u} = g \text{ in } S,

where \( \overline{f}_{xu} = f - f^1 - \beta(k \ast u^1) + \beta(c_k - k(\cdot, 0))u^1 \). By Theorem 1 (ii), problem (41) has a unique solution $u^1 \in C^{2+1,\frac{1}{2}}(Q)$. Observing that $f^1 \geq 0$ and $f^1 \neq 0$ and using the well-known strong extremum principles for parabolic equations (see e.g. Theorem 6.1.1 (ii) in [8]), we deduce the relation

\begin{equation}
\frac{d}{dt} u^1(x, T) > 0 \quad \text{in } \Omega \setminus \{0\} \text{ in case I (II)}.
\end{equation}

Since $\overline{f}_{xu} \in L^p(Q)$, $u_0 \in W^{2, \frac{1}{p}}(\Omega)$ and $g \in W^{2, 1, \frac{1}{p}}(\Omega)$, by Theorem 1 (i) the problem (42) has a unique solution $\overline{u} \in W^{2, 1}(Q)$. Adding (41) and (42) we see that $u^1 + \overline{u}$ solves (13), hence $u = u^1 + \overline{u}$. Observing the assumptions of Theorem and the relations $f^1 + f^t \geq 0$ and $c_k - k(\cdot, 0) \geq 0$, following from the definitions of $f^t$ and $c_k$, we get $\overline{f}_{xu} \geq 0$. Since, in addition, $u_0 \geq 0$ and $g \geq 0$, assertion (i) implies $\overline{u} \geq 0$. Finally, using (43) we prove assertion (ii) for $u = u^1 + \overline{u}$.

Secondly, in case $g \neq 0$ we define a function $g^1 \in C^{\infty}(\overline{S})$ such that $g^1 \geq 0$, $g^1 \neq 0$ and $g^1 \leq g$. Then we set $f^1 = 0$ and replace the boundary conditions of the problems (41) and (42) by $B_1 u^1 = g^i$ and $B_1 u = g - g^i$, respectively. The other arguments concerning the solutions of these systems are almost the same as in the case $f \neq 0$. The proof is complete.

From now on, let the given functions $\beta, a_{ij}, a_j$ and $k$ have again the domains specified in Section 2, i.e. $\beta, a_{ij}, a_j$ depend only on $x$ and $k$ depends only on $t$.

**Remark 1.** For $k$, depending only on $t$, the assumptions (24) read

\begin{equation}
k \in W^1(0, T), \quad k \geq 0, \quad k' \leq 0.
\end{equation}

Let us deduce sufficient conditions for these assumptions in terms of the original kernel $m$. Provided $m \in W^1(0, T)$, the solution $k$ of (12) belongs to $W^1(0, T)$ and has the following representation in the form of Neumann series:

\begin{equation}
k = \sum_{i=0}^{\infty} m(\ast m)^i, \quad k' = \sum_{i=0}^{\infty} m(0)m + m'(\ast m)^i.
\end{equation}

Thus, the sufficient conditions for (44) are

\begin{equation}m \in W^1(0, T), \quad m \geq 0, \quad m'(t) \leq -m(0)m(t).
\end{equation}

For instance, the widely used exponential kernels $m(t) = \sum_{i=1}^{N} \alpha_i e^{-\beta_i t}$ satisfy the conditions (45) provided $\beta_i \geq \alpha_i \geq 0$.

**Remark 2.** Let us return to the direct problem for $u$ in the original form (1), (5). Observing Lemma 1 and Theorem 2, we see that the assumptions for the concentration (or temperature) $u$ to be nonnegative are $f = \varphi + k \ast \varphi \geq 0$, $u_0 \geq 0$ and $g = b \geq 0$ in case I and $g = b + k \ast b \geq 0$ in case II. In case of non-vanishing $k \geq 0$ the conditions $\varphi + k \ast \varphi \geq 0$ and $b + k \ast b \geq 0$ are weaker than the conditions $\varphi \geq 0$ and $b \geq 0$ that occur in the case of vanishing $k$. (The latter case corresponds to the usual parabolic problem without the integral term.) Physically, this phenomenon...
can be explained by a certain inertia of the medium with memory. For instance, choosing the following source term

\[ \chi = 1 \text{ in } \Omega \times (0, T - \delta) \quad \text{and} \quad \chi = -\epsilon < 0 \text{ in } \Omega \times (T - \delta, T), \]

where the numbers \( \epsilon > 0 \) and \( \delta > 0 \) are sufficiently small, so that

\[ \int_0^{T - \delta} k(t - \tau) d\tau \geq \epsilon \left( 1 + \int_{T - \delta}^T k(t - \tau) d\tau \right) \quad \text{for any} \quad t \in (T - \delta, T) \]

we have \( \chi + k * \chi \geq 0 \) in Q that implies \( u \geq 0 \) in Q.

4. Results for IP1. We start by proving a technical result.

**Lemma 2.** Let (3), (4) hold. Assume \( \beta \in C^l(\Omega) \) with some \( l \in (0, 1) \), \( a_{i,j}, a_j \in C(\Omega) \), \( a \in C(\Omega) \), \( a_e \in L^p(Q) \), \( k \in L^p(0, T) \) with some \( p \in (1, \min\{ \frac{2}{\alpha}, \frac{1}{\alpha - 1} \}) \) and the problem (13) has a solution \( u \in W^{2,1}_p(Q) \) such that \( B_1 u \) is continuous in a neighborhood of \( \Gamma \times \{0\} \). If \( f \in L^p(Q), u_0 \in W^{2,1}_p(\Omega), A(0)u_0 + f(\cdot, 0) \in W^{2,1}_p(\Omega) \) and \( g_t \in W^{\infty, \infty}_p(\Omega) \) and \( g_t \in W^{\infty, \infty}_p(\Omega) \) then \( u_t \in W^{2,1}_p(Q) \).

**Proof.** Let us consider the problem

\[ \beta(w_t + k * w_t) = Aw + f_t - k(A(0)u_0 + f(\cdot, 0)) + a_t u \text{ in } Q, \]

\[ v = \frac{1}{\beta}(A(0)u_0 + f(\cdot, 0)) \text{ in } \Omega \times \{0\}, \quad B_1 v = g_t \text{ in } S. \]

Due to the assumptions imposed on \( \beta, A(0)u_0 + f(\cdot, 0) \) and the relation \( p < \frac{2}{\alpha - 1} \) we have \( \frac{1}{\beta}(A(0)u_0 + f(\cdot, 0)) \in W^{2,1}_p(Q) \). Moreover, \( f_t - k(A(0)u_0 + f(\cdot, 0)) + a_t u \in L^p(Q) \). Therefore, by Theorem 1 (i), the problem (46) has a unique solution \( v \) in \( W^{2,1}_p(Q) \). Let us define \( w(x, t) = \int_0^t v(x, \tau) d\tau + u_0(x) \in W^{2,1}_p(Q) \) and integrate the integro-differential equation and boundary condition in (46) with respect time from 0 to \( t \). Taking into account the initial values of \( v \) and \( w \) and the consistency condition \( B_1 u_0 = g(\cdot, 0) \), that holds in view of the continuity assumption of \( B_1 u \), we reach the following relations:

\[ \beta(w_t + k * w_t) = Aw + f_t + f \text{ in } Q, \quad w = u_0 \text{ in } \Omega \times \{0\}, \quad B_1 w = g \text{ in } S, \]

where \( f(x, t) = \int_0^t (\alpha(x, \tau) - a(x, \tau))(u_t(x, \tau) - w_t(x, \tau)) d\tau \). Subtracting (47) from (13) we obtain the problem for \( \omega = u - w \)

\[ \beta(\omega_t + k * \omega_t) = Aw + f \text{ in } Q, \quad \omega = 0 \text{ in } \Omega \times \{0\}, \quad B_1 \omega = 0 \text{ in } S. \]

Using the estimate (17) for the solution of this problem, we obtain \( \|\omega\|_{W^{2,1}_p(Q_t)} \leq C_{(1,0)} \|\bar{f}\|_{L^p(Q_t)} \) for any \( t \in (0, T) \), where \( Q_t = \Omega \times (0, t) \), as before. Further, observing the continuity of \( \alpha \) and the inequality (19) with \( k = 1 \), we deduce

\[ \|\bar{f}\|_{L^p(Q_t)} \leq 2\|a\|_{C(\Omega)} (1 + 1) \|\omega_1\|_{L^p(Q_t)} \]

\[ \leq 2\|a\|_{C(\Omega)} \int_0^t \|\omega_t\|_{L^p(Q_t)} d\tau + 2\|a\|_{C(\Omega)} \int_0^T \|\omega\|_{W^{2,1}_p(Q_t)} d\tau \]

for any \( t \in (0, T) \). Thus, we obtain the following integral inequality:

\[ \|\omega\|_{W^{2,1}_p(Q_t)} \leq \text{Const} \int_0^t \|\omega\|_{W^{2,1}_p(Q_t)} d\tau, \quad t \in (0, T). \]
By Gronwall’s lemma, the solution \( \|w\|_{W^{2,1}_p(Q)} \) of this inequality is zero. Consequently, \( \omega = u - w = 0 \). Since \( v = u_t \in W^{2,1}_p(Q) \), we prove the assertion \( u_t \in W^{2,1}_p(Q) \).

Due to Lemma 1, IP1 is equivalent to the following problem for \((z, u)\):

\[
\beta(u_t + k \ast u_t) = Au + zr + f_0 \quad \text{in } Q, \quad u = u_0 \quad \text{in } \Omega \times \{0\}, \quad B_1 u = g \quad \text{in } S,
\]

where \( B_1, g \) are given by (15), (16) and

\[
r = \phi + k \ast \phi, \quad f_0 = \chi_0 + k \ast \chi_0.
\]

Let us prove the following uniqueness result for the problem (48), (49):

**Theorem 3.** Let (3), (4), (44) hold and \( \beta, a_i, a_j \in C(\Omega), a \in C^{l,1+ \frac{1}{2}}(\Omega), a_i \in L^p(Q) \) with some \( l \in (0, 1), p \in (1, \infty) \). Moreover, let \( a_i \geq 0 \) in \( Q \), \( r \in C^{l,1+ \frac{1}{2}}(\Omega), r_i \in L^p(Q) \),

\[
r \geq 0, \quad r_1 + k \ast r_t - \theta r \geq 0 \quad \text{in } Q
\]

and

\[
\text{(51) for any } U \subseteq \Omega, \text{ meas } U > 0, \text{ it holds } r_1 + k \ast r_t - \theta r \neq 0 \quad \text{in } U \times (0, T).
\]

Here

\[
\theta = \sup_{x \in \Omega} \frac{a(x, T)}{\beta(x)}.
\]

If \((z, u) \in C^l(\Omega) \times C^{2+l,1+ \frac{1}{2}}(Q) \) solves (48), (49) and \( f_0, u_0, g, u_T = 0 \) then \( z = 0, u = 0 \).

**Proof.** Note that (52) immediately implies that

\[
\text{(53) for any } U \subseteq \Omega, \text{ meas } U > 0, \text{ it holds } r \neq 0 \quad \text{in } U \times (0, T).
\]

Suppose contrary that \( z \neq 0 \) and denote \( z^+ = \frac{|z| + z}{2}, \quad z^- = \frac{|z| - z}{2} \).

Firstly, we show that

\[
\text{(54) } z^+ \neq 0 \quad \text{and } \quad z^- \neq 0.
\]

Indeed, let \( z^- = 0 \). Then \( zr = z^+ r \geq 0 \) and due to \( z \neq 0 \) and (53) it holds \( zr \neq 0 \). Observing the assumptions \( f_0, u_0, g = 0 \) and applying Theorem 2 to the solution \( u \) of the problem (48) we get \( u(x, T) > 0, \quad x \in \Omega \). But this contradicts to the assumption \( u_T = 0 \). Similarly, we reach the contradiction in case \( z^+ \neq 0 \) making use of Theorem 2 for \(-u\).

Further, let us consider the following problems for \( u^\pm \):

\[
\beta(u_t^\pm + k \ast u_t^\pm) = Au^\pm + z^\pm r \quad \text{in } Q, \quad u^\pm = 0 \quad \text{in } \Omega \times \{0\}, \quad B_1 u^\pm = 0 \quad \text{in } S.
\]

By the assumptions of theorem, it holds \( z^\pm r \in C^{l,1+ \frac{1}{2}}(Q) \). Moreover, in case I the assumption \( u \in C^{2+l,1+ \frac{1}{2}}(Q) \) implies the consistency condition \( zr = 0 \) in \( \Gamma \times \{0\} \) for the right-hand side of (48). This yields \( z^\pm r = 0 \) in \( \Gamma \times \{0\} \), too. Consequently, assumptions of Theorem 1 (ii) are satisfied for problems (55). Hence, they have the unique solutions \( u^\pm \in C^{2+l,1+ \frac{1}{2}}(Q) \). Let us prove the following inequalities:

\[
\text{(56) } u^\pm \geq 0, \quad u^\pm(\cdot, T) > 0 \quad \text{in } \Omega \quad (\Omega) \text{ in case I (II)},
\]

\[
\text{(57) } u_t^\pm + k \ast u_t^\pm - \theta u^\pm \geq 0, \quad (u_t^\pm + k \ast u_t^\pm - \theta u^\pm)(\cdot, T) > 0 \quad \text{in } \Omega \quad (\Omega) \text{ in case I (II)}.
\]
By (51) and \( z^+ \geq 0 \) we have \( z^+ r \geq 0 \). Moreover, by (53) and (54) we get \( z^+ r \neq 0 \). Using Theorem 2 for solutions of problems (55) we immediately obtain (56). Let us prove (57). Assume without restriction of generality that \( p \in (1, \min \{ \frac{1}{2}, \frac{1}{2m} \}) \). Then the assumptions of Lemma 2 are satisfied for the solutions \( u^\pm \) of the problems (55). Thus, we obtain \( u^+_t \in W^{2,1}_{p}(Q) \). From (55) we deduce the following problems for the functions \( v^\pm = u^+_t + k \cdot u^+_t - \theta u^\pm \in W^{2,1}_{p}(Q) \cap C^{1,1}(\bar{Q}) \):

\[
\begin{align*}
\beta(v^+_t + k \cdot v^+_t) &= A v^+ + z^+ r \kappa + k \cdot r_t - \theta r | + f^+_t \quad \text{in } Q, \\
v^+ &= \frac{1}{\beta} z^+ r \quad \text{in } \Omega \times \{0\}, \quad B_t v^+ = 0 \quad \text{in } S,
\end{align*}
\]

where

\[
f^+_t(x, t) = a(x, t) u^+_t(x, t) + \int_0^t k'(t - \tau)(a(x, \tau) - a(x, t)) u^+_t(x, \tau) \, d\tau.
\]

By the assumptions of theorem and (56), (54) we have \( z^+ r \kappa + k \cdot r_t - \theta r T + f^+_T \geq 0 \) and \( z^+ |r + k \cdot r_t - \theta r T + f^+_T \neq 0 \). Consequently, Theorem 2 implies (57).

Let \( x^* \in \overline{\Omega} \) be such that

\[
u^-(x, T) \leq u^+(x^*, T) \quad \text{for any } x \in \overline{\Omega}.
\]

Since \( u = u^+ - u^- \) and \( u(x, T) = u_T(x) \equiv 0 \), we have \( u^+(x, T) = u^-(x, T), x \in \overline{\Omega} \), and (59) implies

\[
u^-(x, T) \leq u^-(x^*, T) \quad \text{for any } x \in \overline{\Omega}.
\]

Let us show that the point \( x^* \) is the stationary maximum of \( u^+(x, T), \) i.e.

\[
\nabla x \cdot u^+(x^*, T) = 0.
\]

The equality (61) may fail only when \( x^* \in \Gamma \). In case I we have the boundary condition \( u = 0 \) in \( \Gamma \), hence in view of (56) the function \( u^+(x, T) \) cannot achieve its maximum on \( \Gamma \), and we automatically get (61). It remains to show (61) for the case II when \( x^* \in \Gamma \). In this case \( \omega(x^*) \cdot \nabla x \cdot u^+(x^*, T) = 0 \), where \( \omega(x^*) \) is an outer direction at \( x^* \). Furthermore, since \( u^+(x, T) \) achieves its maximum over \( \Gamma \) in the point \( x = x^* \), we have \( \tau \cdot \nabla x \cdot u^+(x^*, T) = 0 \), where \( \tau \) is any tangential direction at \( x^* \). Summing up, we get \( \xi \cdot \nabla x \cdot u^+(x^*, T) = 0 \), where \( \xi \) is any direction. This yields (61).

By the definition of \( z^+ \) and \( z^- \), it holds either \( z^+(x^*) = 0 \) or \( z^-(x^*) = 0 \). In case \( z^+(x^*) = 0 \) we have \( z^+(x^*) = 0 \) and from the equation (55) we obtain

\[
[\beta(u^+_t + k \cdot u^+_t) - au^+] (x^*, T) = A_0 u^-(x^*, T), \quad A_0 = A - a.
\]

The left-hand side of (62) is strictly positive due to the inequalities (4), (56) and (57). Indeed:

\[
[\beta(u^+_t + k \cdot u^+_t) - au^+] (x^*, T) = \beta(x^*) [u^+_t + k \cdot u^+_t - \frac{a}{\beta} u^+](x^*, T) \geq \beta_0 [u^+_t + k \cdot u^+_t - \theta u^+](x^*, T) > 0.
\]

Therefore, (63)

\[
Au^+(x^*, T) > 0.
\]
On the other hand, since \( x = x^* \) is the stationary maximum point of \( u^+(x, T) \) and the principal part of \( A_0 \) is elliptic (see (3)), we obtain

\[
A_0 u^+(x^*, T) = \sum_{i,j=1}^n a_{ij}(x^*) u_{x_i x_j}^+(x^*, T) + \sum_{j=1}^n a_j(x^*) u_{x_j}^+(x^*, T) \leq 0.
\]

This contradicts (63). Analogously we come to a contradiction in case \( z^-(x^*) = 0 \). Hence, the assumption \( z \neq 0 \) was incorrect. We have \( z = 0 \). Finally, whereas \( f_0, u_0, g = 0 \) by assumption, problem (48) is homogeneous. Thus, by Theorem 1 it holds \( u = 0 \). Proof is complete.

**Remark 3.** We note that in case \( k \) is non-vanishing and satisfies (44) the conditions (51) with (50) for the coefficient \( \phi \) are weaker than in case \( k = 0 \) when the usual parabolic problem occurs.

To deal with the existence and stability issues, we have to impose additional assumptions on \( r \):

\[
(64) \quad r \geq \delta \quad \text{in } \Omega \times (T - \delta, T) \quad \text{with some } \delta \in (0, \frac{T}{2}) \quad \text{and}
\]

either \( r \geq \delta \) in \( \Omega \times (0, \delta) \) (case (1)) or \( r = 0 \) in \( \Omega \times (0, \delta) \) (case (2)).

Note that in case I & (1) the following explicit relation for boundary values of \( z \) from (48) can be obtained:

\[
(65) \quad z = r^{-1}(g_t - A u_0 - f) \quad \text{in } \Gamma \times \{0\}.
\]

In this case it is possible to reduce the inverse problem to the case when \( z = 0 \) in \( \Gamma \).

Indeed, let us represent the term \( z r \) in (48) in the form \( z r = (z - \bar{z}) r + \bar{z} r \), where \( \bar{z} \) is an arbitrary function satisfying the condition \( \bar{z} = r^{-1}(g_t - A u_0 - f) \) in \( \Gamma \times \{0\} \), shift the addend \( \bar{z} r \) into \( f_0 \) and consider the inverse problem for the unknown \( z - \bar{z} \) in place of \( z \). Then this new unknown is zero in \( \Gamma \).

Now we are going to prove an existence and stability theorem for (48), (49). The Fredholm-type result of this theorem (i.e. the assertion (i) below) was already obtained in [11], but under different assumptions. Namely, in [11] certain positivity conditions on the original kernel \( m \) where imposed. We do not need such assumptions in the assertion (i).

**Theorem 4.** Let (3), (4) hold and \( \beta, a_{ij}, a_j, \beta \in C^l(\Omega), a \in C^{l+\frac{1}{2}}(Q), a_t \in L^p(Q), \) with some \( l \in (0, 1), p \in (1, \infty) \). Moreover, let \( a_t \geq 0, r \in C^{l+\frac{1}{2}}(Q), r_t \in L^p(Q) \) and (64) hold. In addition, let \( f \in C^{l+\frac{1}{2}}(Q), u_0 \in C^{l+1}(\Omega), g \in C^{2-l-\frac{1}{2}}(S), w_t \in C^{2+l}(\Omega) \) and the consistency conditions

\[
(66) \quad u_0 = g, \quad \beta g_t = A u_0 + f_0 \quad \text{in case I,} \quad \omega \cdot \nabla_x u_0 = g \quad \text{in case II in } \Gamma \times \{0\}
\]

be satisfied. Then the following assertions are valid.

(i) If \( k \in W^{1+\frac{1}{2}}_{p, r} (0, T) \), \( r \geq 0, r_t - \theta r \geq 0, \)

(67) for any \( U \subseteq \Omega, \text{ meas } U > 0, \) it holds \( r_t - \theta r \neq 0 \) in \( U \times (0, T) \).
and the homogeneous inverse problem, i.e.

\[ \beta(u_0^2 + k * v_0^2) = Av^0 + q^0 r \quad \text{in} \quad Q, \]
\[ v^0 = 0 \quad \text{in} \quad \Omega \times \{0\}, \quad B_1 v^0 = 0 \quad \text{in} \quad S, \quad v^0 = 0 \quad \text{in} \quad \Omega \times \{T\} \]

has in \( C^{4}(\Omega) \times C^{2+\frac{1}{4} + \frac{r}{4}}(Q) \) only the trivial solution \( q^0 = 0, v^0 = 0 \) then the inverse problem (48), (49) has a unique solution \((z, u)\) in the space \( C^{4}(\Omega) \times C^{2+\frac{1}{4} + \frac{r}{4}}(Q) \) and in case I & (1) it holds \( z = 0 \) in \( \Gamma \). Moreover, the solution \((z, u)\) satisfies the estimate

\[ \|z\|_1 + \|u\|_{2+\frac{r}{4} + \frac{1}{4}} \leq \Lambda(\beta, a_{ij}, a_j, a, k, r) \times \left\{ \|f_0\|_{\frac{r}{4}} + \|u_0\|_{2+\frac{r}{4}} + \|q\|_{2+\frac{r}{4} + \frac{1}{4} + \frac{r}{4}} + \|u_T\|_{2+\frac{r}{4}} \right\} \]

with some constant \( \Lambda \) depending on the quantities shown in brackets.

(ii) If

\[ k \in W^{1, \infty}_{\text{loc}}(0, T), \quad k \geq 0, \quad k' \leq 0 \]

and \( r \) satisfies (51), (52) then the inverse problem (48), (49) has a unique solution \((z, u)\) in the space \( C^{4}(\Omega) \times C^{2+\frac{1}{4} + \frac{r}{4}}(Q) \) and in case I & (1) it holds \( z = 0 \) in \( \Gamma \). The solution satisfies the estimate (69).

Proof. Evidently, the assertion (ii) follows from the assertion (i) and Theorem 3. Therefore, let us prove (i). We note that, due the assumptions of the theorem, the inverse problem in case \( k = 0 \), i.e. the problem

\[ \beta u_1^2 = Au^1 + z^1 r + f_0 \quad \text{in} \quad Q, \]
\[ u^1 = u_0 \quad \text{in} \quad \Omega \times \{0\}, \quad B_1 u^1 = g \quad \text{in} \quad S, \quad u^1 = u_T \quad \text{in} \quad \Omega \times \{T\} \]

has a unique solution \((z^1, u^1) \in \mathcal{X} = C^{4}(\Omega) \times C^{2+\frac{1}{4} + \frac{r}{4}}(Q) \) where

\[ C^{4}(\Omega) = \{ z : z \in C^{4}(\Omega), \quad z = 0 \quad \text{in} \quad \Gamma \quad \text{in case I & (1)} \}, \]

and this solution satisfies the estimate

\[ \|z^1\|_1 + \|u^1\|_{2+\frac{r}{4} + \frac{1}{4}} \leq \Lambda^{2}(\beta, a_{ij}, a_j, a, r) \times \left\{ \|f_0\|_{\frac{r}{4}} + \|u_0\|_{2+\frac{r}{4}} + \|q\|_{2+\frac{r}{4} + \frac{1}{4} + \frac{r}{4}} + \|u_T\|_{2+\frac{r}{4}} \right\}. \]

This assertion follows from a known Fredholm-type result for the problem (71) (Theorem 1.2 in [7]) and the related uniqueness result (Theorem 3 of the present paper with \( k = 0 \)).

Let us denote \( q = z - z^1 \) and \( v = u - u^1 \). Then the inverse problem (48), (49) for \((z, u) \in \mathcal{X} \) is equivalent to the following inverse problem for the pair \( X = (q, v) \in \mathcal{X} \):

\[ \beta v_1 = Av + qr - \beta k * (u_1^2 + v_1) \quad \text{in} \quad Q, \]
\[ v = 0 \quad \text{in} \quad \Omega \times \{0\}, \quad B_1 v = 0 \quad \text{in} \quad S, \quad v = 0 \quad \text{in} \quad \Omega \times \{T\} \]

Let \( \mathcal{P} \) stand for the operator that assigns to a given right-hand side \( f_0 \) the solution of the problem (71) with \( u_0, g, u_T = 0 \). Then the problem (73) is in \( \mathcal{X} \) equivalent to the operator equation

\[ X = TX + \Psi \]

\[ ^1 \text{The uniqueness for (71) was proved also in [7], but under different assumptions.} \]
where $T(v, u) = \mathcal{P}(-\beta k + u) = \mathcal{P}(-\beta k' * v - \beta k(0) v)$ and $\Psi = \mathcal{P}(-\beta k * u_k')$.

Observing the assumptions of theorem and the implication $k' \in L^{\frac{1}{q'}}(0, T), w \in C^{\frac{1}{q'}}(Q) \Rightarrow k' * w \in C^{\frac{1}{q'}}(Q)$, following from (23), we see that $T \in \mathcal{L}(\mathcal{U}^i, \mathcal{X}^i)$ and $\Psi \in \mathcal{X}^i$, where $\mathcal{U}^i = C^i(\Omega) \times C^{1+i}(\Omega)$ and $\mathcal{X}^i$ is an arbitrary number in the interval $(0, l)$. Since $\mathcal{X}^i$ is compactly embedded in $\mathcal{U}^i$, the operator $T$ is compact in $\mathcal{U}^i \times \mathcal{X}^i$. Moreover, 1 is not an eigenvalue of $T$, because the equation $X^0 = T X^0$ is in $\mathcal{X}^i$ equivalent to the problem (68), whose solution $X^0 = (q^0, v^0)$ is zero by the assumption. Consequently, by the Fredholm’s alternative, the equation (74) has a unique solution in $\mathcal{X}^i$. This proves the existence assertion of theorem.

It remains to prove (69). Since 1 belongs to the resolvent set of $T$, the operator $T$ depends on $\lambda(a_{ij}, a_{j}, a_{k}, r)$ and the inequality $\| \cdot \|_{\mathcal{U}^i, \mathcal{X}} \leq \text{Const} \| \cdot \|_{\mathcal{X}^i}$ holds, we get

$$\| X \|_{\mathcal{U}^i, \mathcal{X}} \leq \Lambda^2(\beta, a_{ij}, a_{j}, a_{k}, r) \| \Psi \|_{\mathcal{U}^i, \mathcal{X}} \leq \Lambda^2(\beta, a_{ij}, a_{j}, a_{k}, r) \| \Psi \|_{\mathcal{X}^i}.$$  

Let us estimate the equation (74) taking the relations (75), (23) and (72) into account:

$$\| g \|_{L^2} \| v \|_{L^2} \leq \| X \|_{\mathcal{X}^i} \leq \| T \| \| X \|_{\mathcal{U}^i, \mathcal{X}} + \| \Psi \|_{\mathcal{X}^i} \leq \Lambda^2(\beta, a_{ij}, a_{j}, a_{k}, r) \| T \| \| \Psi \|_{\mathcal{X}^i} \leq \Lambda^2(\beta, a_{ij}, a_{j}, a_{k}, r) \left\{ \| g \|_{L^2} + \| u_0 \|_{L^2} + \| u_0 \|_{L^2} + \| u_\tau \|_{L^2} \right\}.$$  

Since $z = z^1 + q$ and $u = u^1 + v$, the estimate (69) follows from (72) and (76).

Let us denote by $\tilde{F}_{\beta, a, r}$ the operator that assigns to the vector $d = (f_0, u_0, g, u_\tau)$ the solution of the linear inverse problem (48), (49). Provided $\beta, a, r$ and also $a_{ij}, a_{j}, k$ satisfy the assumptions of Theorem 4 (incl. the additional assumptions of the assertion (ii) of Theorem 4), the operator $\tilde{F}_{\beta, a, r}$ is well-defined from the space

$$\mathcal{D} = \{d: d \in C^1(\Omega) \times C^{2+i}(\Omega) \times C^{2+i-\nu, 1+i-\nu}(S) \times C^{2+i}(\Omega), d \text{ satisfy the consistency conditions (66)}\}$$

to the space $C^i(\Omega) \times C^{2+i, 1+i}(Q)$ and satisfies the estimate

$$\| \tilde{F}_{\beta, a, r}(f_0, u_0, g, u_\tau) \|_{C^i(\Omega) \times C^{2+i, 1+i}(Q)} \leq \Lambda(\beta, a_{ij}, a_{j}, a_{k}, r) \left\{ \| f_0 \|_{L^2} + \| u_0 \|_{L^2} + \| g \|_{L^2} + \| u_\tau \|_{L^2} \right\}.$$  

5. Results for IP2 and IP3. By Lemma 1, IP2 is equivalent to the following problem for $(a, u)$:

$$\beta(u_t) + k * u_t = A_0 u + u + f \quad \text{in} \ Q, \quad u = u_0 \quad \text{in} \ \Omega \times \{0\}, \quad B_1 u = g \quad \text{in} \ S,$$

$$u = u_\tau \quad \text{in} \ \Omega \times \{T\},$$

where $f$, $B_1$ and $g$ are given by (14) - (16) and $A_0 u = \sum_{i,j} a_{ij} u_{x_i x_j} + \sum_{i,j} \alpha_{ij} u_{x_j}$.

Let us define the following set of the coefficients $\alpha$ that depends on $\theta \in \mathbb{R}$:

$$\mathcal{A}_{\theta} = \{a \in C^i(\Omega) : \sup_{x \in \Omega} \frac{\alpha(x)}{\beta(x)} \leq \theta\}.$$  

We will prove a theorem that comprises global uniqueness and local conditional existence and stability.
Theorem 5. Let (3), (4) hold, \(\beta, a_{ij}, a_j \in C^1(\Omega)\) with some \(l \in (0, 1)\) and \(\theta \in \mathbb{R}\). Then the following assertions are valid.

(i) If \(k\) satisfies (44) and the problem (78), (79) has the solutions \((a_1, u_1) \in C^1(\Omega) \times C^{2+l, 1+\frac{\nu}{2}}(Q), (a_2, u_2) \in A^{\beta, \theta}_{a_1} \times C^{2+l, 1+\frac{\nu}{2}}(Q)\), where \(u = u_1\) satisfies the conditions

\[
u \geq 0, \quad u_t + k \ast u_t - \theta u \geq 0,
\]

for any \(U \subseteq \Omega, \text{meas} \ U > 0\), it holds \(u_t + k \ast u_t - \theta u \neq 0\) in \(U \times (0, T)\),

then \(a_1 = a_2\) and \(u_1 = u_2\).

(ii) If \(k\) satisfies (70) and (78), (79) has a solution \((a, u) \in A^{\beta, \theta}_{a} \times C^{2+l, 1+\frac{\nu}{2}}(Q)\) such that \(u\) fulfills (80).

\[
\begin{align*}
u \geq \delta \quad \text{in} \quad \Omega \times (T - \delta, T) \quad \text{and} \quad u = 0 \quad \text{in} \quad \Omega \times (0, \delta) \quad \text{with} \quad \delta \in (0, \frac{T}{2}),
quasi-b \begin{cases} \mathcal{F} := \left\| f - f \right\|_{L^2}^2 + \left\| \overline{u}_0 - u_0 \right\|_{2+l}^2 + \left\| \overline{g} - g \right\|_{2+l, \nu, 1+\frac{\nu}{2}}^2 + \left\| \overline{u}_T - u_T \right\|_{2+l}^2 < \frac{1}{2\lambda^2}, & \text{where} \quad \lambda = \Lambda(\beta, a_{ij}, a_j, a, k, u), \\
\overline{u}_0 = \overline{g}, \quad \overline{u}_T = \overline{g} & \text{in case I}, \\
\overline{u}_0 = \overline{g}, \quad \overline{u}_T = \overline{g} & \text{in case II in} \quad \Gamma \times \{0\},
quasi-c \begin{cases} \overline{u}_T = \overline{g} & \text{in case I,} \\
\overline{u}_T = \overline{g} & \text{in case II in} \quad \Gamma \times \{T\},
quasi-d \begin{cases} \overline{u}_0 = u_0 & \text{in case I in} \quad \Gamma, \text{the problem (78), (79) with} \quad f_0, u_0, g, u_T \text{replaced by} \quad f_0, u_0, g, u_T \text{has a unique solution} \quad (\overline{u}, \overline{u}) \text{in the ball} \quad U \quad \text{where} \quad U = \left\{ (\overline{u}, \overline{u}) : \left\| \overline{u} - a \right\| + \left\| \overline{u} - u \right\|_{2+l, 1+\frac{\nu}{2}} < \frac{1}{\lambda} \left( 1 - \sqrt{1 - 2\lambda^2} \right) \right\}.
quasi-e \begin{cases} \overline{u} = u_0, \quad \beta \gamma_t + A_0 u_0 + \theta \overline{u}_0 = f & \text{in case I}, \\
\overline{u} = u_0, \quad \beta \gamma_t + A_0 u_0 + \theta \overline{u}_0 = f & \text{in case II in} \quad \Gamma \times \{0\},
quasi-f \begin{cases} \overline{u}_0 \geq 0, \quad \overline{u}_0 \geq 0, \quad f_t + k \ast f_t - \theta f \geq 0, \quad g_t + k \ast g_t - \theta g \geq 0, \\
f_t + k \ast f_t - \theta f \neq 0 \text{ or } \quad g_t + k \ast g_t - \theta g \neq 0 \text{ and} \quad (\theta \beta - a) \overline{u}_0 \leq A_0 u_0 + f(\cdot, 0)
quasi-g \begin{cases} \text{then the solution} \quad u \text{ of the direct problem (78) belongs to} \quad C^{2+l, 1+\frac{\nu}{2}}(Q) \text{ and satisfies (80). If, in addition,} \quad f(\cdot, t) = 0 \quad \text{and} \quad g(\cdot, t) = 0 \text{ for} \quad t \in (0, \delta_t) \text{ with some} \quad \delta_t \in (0, \frac{T}{2}), \quad u_0 = 0 \quad \text{and} \quad g > 0 \text{ in} \quad \Gamma \times \{T\} \text{ in case I, then} \quad u \text{ satisfies (81), too.}}
quasi-h \begin{cases} \text{Remark 4.} \\
1. \text{Since} \quad \frac{1}{\lambda} \left( 1 - \sqrt{1 - 2\lambda^2} \right) = \lambda D \text{ as} \quad D \to 0^+, \text{the relation (85) implies that the solution operator of the problem (78), (79) is locally Lipschitz-continuous in the neighborhood of} \quad (a, u).\end{cases}
\end{cases}
\end{align*}

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2. In case \( u_0 = 0 \) the assumptions of (iii) do not contain \( a \), except for the condition \( a \in A_{\beta}^{\alpha} \). Namely, in this case (88) is dropped and the consistency conditions (86) read \( g = 0 \), \( \beta \eta = 0 \) in case I and \( g = 0 \) in case II in \( \Gamma \times \{ 0 \} \).

Proof of Theorem 5. Subtracting the problems (78), (79) for the pairs \((a_1, u_1)\) and \((a_2, u_2)\) we obtain the problem (48), (49) for the difference \( z = a_1 - a_2, u = u_1 - u_2 \) that has the zero free term \( f_0 \), zero initial, boundary and final conditions and contains \( a_2 \) and \( u_2 \) instead of \( a \) and \( r \), respectively. Note that the conditions (89) hold with \( \theta \) replaced by \( \theta_2 = \sup_{x \in \Omega} \frac{\omega(x)}{\rho(x)} \leq \theta \), too. Thus, we can apply Theorem 3 to the problem for \( z \) and \( u \) to obtain (i).

Let us prove (ii). Firstly, we note that the problem for \((\overline{a}, \overline{u})\) is equivalent to the following problem for the differences \( z = \overline{a} - a, w = \overline{u} - u:\)

\[
\begin{align*}
\beta(w_t + k \cdot u) &= (A_0 + a)w + zu + zw + \overline{f} - f \quad \text{in} \, Q, \\
w &= \overline{u}_0 - u_0 \quad \text{in} \, \Omega \times \{ 0 \}, \\
B_1 w &= \overline{g} - g \quad \text{in} \, S, \\
w &= \overline{u}_T - u_T \quad \text{in} \, \Omega \times \{ T \}.
\end{align*}
\]

Further, owing to the properties of \( \beta, a, u \) and also \( a_1, a_2, k \), the operator \( \tilde{F}_{3,a,u} \) is well-defined from \( \mathcal{D} \) to \( C_t(\Omega) \times C_{2+l,1+\frac{\lambda}{2}}(Q) \) (definitions of \( \tilde{F} \) and \( \mathcal{D} \) can be found at the end of Section 4). Since \( (zw + \overline{f} - f, \overline{u}_0 - u_0, \overline{g} - g, \overline{u}_T - u_T) \in \mathcal{D} \) for any \( S = (z, w) \in \mathcal{S} \) where

\[
\mathcal{S} = \{ S : S \in C_t(\Omega) \times C_{2+l,1+\frac{\lambda}{2}}(Q), \ w = 0 \ \text{in case I in} \ \Gamma \times \{ 0 \} \}.
\]

we can define the operator \( \tilde{F}(S) = \tilde{F}_{3,a,u}(zw + \overline{f} - f, \overline{u}_0 - u_0, \overline{g} - g, \overline{u}_T - u_T) \) for any \( S \in \mathcal{S} \). Moreover, it holds \( \tilde{F}(\mathcal{S}) \subseteq \mathcal{S} \). Now we can immediately check that the problem (89) is in the space \( \mathcal{S} \) equivalent to the fixed-point equation \( S = \tilde{F}(S) \). Defining \( ||S|| = ||z||_2 + ||w||_{2+l,1+\frac{\lambda}{2}} \) and using the formula (77) and the definitions of \( D \) and \( \lambda \) in (82) we estimate:

\[
\| \tilde{F}(S) \| \leq \lambda \left\{ ||zw||_{l,\frac{\lambda}{2}} + D \right\} \leq \lambda \left\{ ||z||_2 ||w||_{2+l,1+\frac{\lambda}{2}} + D \right\} \leq \lambda \left\{ \frac{1}{2} ||S||^2 + D \right\}.
\]

Similarly, for \( S^j = (z^j, w^j), j = 1, 2, \) in view of the relation

\[
z^1 - z^2 = \frac{z^1 + z^2}{2}(w^1 - w^2) + (z_1 - z_2) \frac{w^1 + w^2}{2}
\]

we obtain

\[
\| \tilde{F}(S^1) - \tilde{F}(S^2) \| = \| \tilde{F}_{3,a,u}(z^1 w^1 - z^2 w^2, 0, 0, 0) \| \leq \lambda \left\{ \frac{S^1 + S^2}{2} \right\} \| S^1 - S^2 \|.
\]

Thus, by virtue of the assumed inequality \( D < \frac{\lambda}{2\sqrt{\pi}} \), for any \( S, S^1, S^2 \in \mathcal{U}_0 \), where

\[
\mathcal{U}_0 = \left\{ S : S \in \mathcal{S}, \ ||S|| < \frac{1}{\lambda} \left( 1 - \sqrt{1 - 2\lambda^2 D} \right) \right\},
\]

we deduce the estimate \( \| \tilde{F}(S) \| < \lambda \left\{ \frac{1}{2} D + 1 \right\} = \rho \) that implies the relation \( \tilde{F}(\mathcal{U}_0) \subseteq \mathcal{U}_0 \) and the inequality \( \| \tilde{F}(S^1) - \tilde{F}(S^2) \| \leq q \| S^1 - S^2 \| \), where \( q = \lambda \rho < 1 \). Therefore, by the contraction principle, the equation \( S = \tilde{F}(S) \) has a unique solution in the ball \( \mathcal{U}_0 \). This proves (ii).

It remains to prove (iii). Applying Theorems 1 and 2 to (78) we immediately get

\[
\begin{align*}
\rho \in C_{2+l,1+\frac{\lambda}{2}}(Q), \\
u \geq 0, \\
u(\cdot, T) > 0 \ \text{in} \ \Omega \ \text{in case I (II)}.
\end{align*}
\]
Further, let us assume without restricting the generality that \( p \in (1, \min\left(\frac{2}{3}, \frac{2}{1+	heta}\right)) \). Then the assumptions of Lemma 2 are satisfied for the solution \( u \) of (78). Applying this lemma, we obtain \( u_t \in W^{2,1}_p(\Omega) \). Thus, the function \( v = u_t + k \cdot u_t - \theta u \) belongs to \( C^{1+\frac{1}{3}}(\Omega) \times W^{2,1}_p(\Omega) \). One can immediately check that \( v \) solves the problem

\[
\beta(u_t + k \cdot u_t) = A_0 u + av + f_t + k \cdot f_t - \theta f \quad \text{in} \quad \Omega,
\]

\[
v = \frac{1}{\beta} [A_0 u_0 + f + (a - \theta \beta) u_0] \quad \text{in} \quad \Omega \times \{0\}, \quad B_1 v = g_t + k \cdot g_t - \theta g \quad \text{in} \quad S.
\]

Observing (87), (88) and Theorem 2 we obtain \( v \geq 0 \), \( v(x,T) > 0 \) for \( x \in \Omega \). These relations with (90) imply (80). Finally, the relations (90) with the additional assumptions \( g > 0 \) in \( \Gamma \times \{T\} \) in case I and \( f(\cdot, t) = 0 \), \( g(\cdot, t) = 0 \) for \( t \in (0,\delta_0) \) with some \( \delta_0 \in (0, \frac{T}{2}) \) and \( u_0 = 0 \) imply (81) with some \( \delta \leq \delta_0 \).

Finally, we study IP3. Due to Lemma 1, IP3 is equivalent to the following problem for \( (\beta, u) \):

(91) \quad \beta(u_t + k \cdot u_t) = A u + f \quad \text{in} \quad \Omega, \quad u = u_0 \quad \text{in} \quad \Omega \times \{0\}, \quad B_1 u = g \quad \text{in} \quad S,

(92) \quad u = u_T \quad \text{in} \quad \Omega \times \{T\},

where \( f, B_1 \) and \( g \) are given by (14) - (16).

Let us introduce the following set for the coefficients \( \beta \) that depends on \( \theta_0 > 0 \):

\[
B_{\beta_0}^I = \{ \beta \in C^1(\Omega) : \inf_{x \in \Omega} \beta(x) \geq \beta_0 \}
\]

and define \( \theta_{\beta_0} = \max\left\{ 0; \frac{1}{\beta_0} \sup_{x \in \Omega} a(x, T) \right\} \). Then we have \( \sup_{x \in \Omega} \frac{a(x, T)}{\beta(x)} \leq \theta_{\beta_0} \) for any \( \beta \in B_{\beta_0}^I \).

Theorem 6. Let (3) hold, \( a_{ij}, a_j \in C^1(\Omega) \), \( a \in C^{1+\frac{1}{3}}(\Omega) \), \( a_i \in L^p(\Omega) \) with some \( l \in (0,1) \), \( p \in (1, \infty) \), \( \alpha_i \geq 0 \) and \( \beta_0 > 0 \). Then the following assertions are valid.

(i) If \( k \) satisfies (44), the problem (91), (92) has the solutions \( (\beta_1, u_1) \in C^1(\Omega) \times C^{2+1+\frac{1}{3}}(\Omega) \) and \( (\beta_2, u_2) \in B_{\beta_0}^I \times C^{2+1+\frac{1}{3}}(\Omega) \) where \( u = u_1 \) satisfies the conditions

\[
\begin{align*}
&u_{tt} \in L^p(\Omega), \quad u_t + k \cdot u_t \geq 0, \\
&\dot{u} := (u_t + k \cdot u_t)_t + k \cdot (u_t + k \cdot u_t)_t - \theta_{\beta_0}(u_t + k \cdot u_t) \geq 0, \\
&\text{for any} \ U \subseteq \Omega, \text{meas} U > 0, \text{it holds} \ \dot{u} \neq 0 \ \text{in} \ U \times (0,T),
\end{align*}
\]

then \( \beta_1 = \beta_0 \) and \( u_1 = u_2 \).

(ii) If \( k \) satisfies (70), \( A u_0 - f = 0 \) in case I in \( \Gamma \times \{0\} \) and the problem (91), (92) has a solution \( (\beta, u) \in B_{\beta_0}^I \times C^{2+1+\frac{1}{3}}(\Omega) \) such that \( u \) fulfills (93),

\[
u_t + k \cdot u_t \geq \delta \quad \text{in} \quad \overline{\Omega} \times (T - \delta, T) \quad \text{and}
\]

\[
u_t = 0 \quad \text{in} \quad \overline{\Omega} \times (0, \delta) \quad \text{with some} \ \delta \in (0, \frac{T}{2}),
\]

then for any \( \tilde{f}, \overline{u}_0, \overline{g}, \overline{u}_T \) such that

\[
\begin{align*}
D < \frac{1}{2\bar{\lambda}^2(1 + \|k\|)} & \quad \text{where} \quad \bar{\lambda} = \Lambda(\beta, a_{ij}, a_j, a, k, u_t + k \cdot u_t), \\
\|k\| & = \|k\|_{C[0,T]} \quad \text{and} \ D \text{ is defined by} \ (82),
\end{align*}
\]
\( \tilde{u}_0 = \tilde{g} \), \( \beta \tilde{u}_t = \Lambda \tilde{u}_0 + \tilde{f} = 0 \) in case I,
\[ \omega \cdot \nabla_x \tilde{u}_0 = \tilde{g} \] in case II in \( \Gamma \times \{0\} \),
and (84) holds, the problem (91), (92) with \( f_0, u_0, g, u_T \) replaced by \( f_0, \tilde{u}_0, \tilde{g}, \tilde{u}_T \) has a unique solution \( (\tilde{\beta}, \tilde{u}) \) in the ball
\[
\tilde{U} = \left\{ (\tilde{\beta}, \tilde{u}) : \|\tilde{\beta} - \beta\|_t + \|\tilde{u} - u\|_{2+\frac{1}{2}+\frac{1}{2}+}\leq \frac{1}{\lambda(1+\|k\|)} \left(1 - \sqrt{1 - 2\lambda^2(1+\|k\|)^2}\delta \right) \right\}.
\]

(iii) If \( k \) satisfies (44), \( \beta \in \mathcal{B}_k^l \), \( \alpha_t = 0 \), \( u_0 \in C^{2+\frac{1}{2}}(\Omega) \), \( A(0)u_0 \in W_p^{2+\frac{1}{2}}(\Omega), \)
\( f \in C^{\frac{1}{2}}(Q), f_t, f_{tt} \in L^p(Q), f_t(\cdot, 0) \in W_p^{2+\frac{1}{2}}(\Omega), g \in C^{2+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}}(S), \)
\( g_t, g_{tt} \in W_p^{2+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}}(S), \)
\( r_f := f_t + k \ast f_t \geq 0, \)
\( \tilde{r}_f := r_{f, t} + k \ast r_{f, t} - \theta_{\beta_0} r_f \geq 0, \)
\( \tilde{r}_g := r_{g, t} + k \ast r_{g, t} - \theta_{\beta_0} r_g \geq 0, \)
\( \tilde{r}_f \neq 0 \) or \( \tilde{r}_g \neq 0, \)
the consistency conditions (86) and the relations
\[
\frac{1}{\beta}(A(0)u_0 + f(\cdot, 0)) \in W_p^{2}(\Omega), A(0)\left[\frac{1}{\beta}(A(0)u_0 + f(\cdot, 0))\right] \in W_p^{2+\frac{1}{2}}(\Omega),
\]
\[
A(0)u_0 + f(\cdot, 0) \geq 0,
\]
\[
A(0)\left(\frac{1}{\beta}(A(0)u_0 + f(\cdot, 0))\right) - \theta_{\beta_0} A(0)u_0 + f_t(\cdot, 0) - \theta_{\beta_0} f(\cdot, 0) \geq 0
\]
hold, then the solution \( u \) of the direct problem (91) belongs to \( C^{2+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}}(Q) \) and satisfies (93). If, in addition,
\[
f_t(\cdot, t) = 0, g_t(\cdot, t) = 0 \text{ for } t \in (0, \delta_0) \text{ with some } \delta_0 \in (0, T),
\]
\[
A(0)u_0 + f(\cdot, 0) = 0
\]
and \( r_g > 0 \) in \( \Gamma \times \{T\} \) in case I, then \( u \) satisfies (94), too.

Remark 5.
1. The relation (97) implies that the solution operator of the problem (91), (92) is locally Lipschitz-continuous in the neighborhood of \((\beta, u)\).
2. If \( u_0 = 0 \) and \( f(\cdot, 0) = 0 \) then the assumptions of (iii) do not contain \( \beta \), except for \( \beta \in \mathcal{B}_k^l \). Namely, the assumptions (99) are dropped and the consistency conditions (86) read \( g(\cdot, 0) = 0, g_t(\cdot, 0) = 0 \) in case I and \( g(\cdot, 0) = 0 \) in case II.

Proof of Theorem 6. Subtracting the problems (91), (92) with the pairs \((\beta_1, u_1)\) and \((\beta_2, u_2)\) we obtain the problem (48), (49) for the difference \( z = \beta_1 - \beta_2, u = u_1 - u_2 \) that has the zero free term \( f_0 \), zero initial, boundary and final conditions and contains \( \beta_2 \) and \( u_{1, t} + k \ast u_{1, t} \) instead of \( \beta \) and \( r \), respectively. Applying Theorem 3 to this problem, we immediately deduce (i).

Let us prove (ii). The problem for \((\tilde{\beta}, \tilde{u})\) is equivalent to the following problem for the differences \( z = \beta - \tilde{\beta}, w = \tilde{u} - u:\)
\[
\beta(z_t + k \ast z_t) = \Lambda w + z(u_t + k \ast u_t) + z(u_t + k \ast u_t) + \tilde{f} - f \text{ in } Q,
\]
\[
w = \tilde{u}_0 - u_0 \text{ in } \Omega \times \{0\}, B_1w = \tilde{g} - g \text{ in } S, w = \tilde{u}_T - u_T \text{ in } \Omega \times \{T\}.
\]
Further, denoting $S = (z, w)$, the problem \((101)\) is in the space
\[\mathcal{J} = \{ S : S \in C^1(\overline{\Omega}) \times C^{2+1,1+\frac{1}{2}}(Q), \; w_t = 0 \text{ in } I \times \{0\}\},\]
equivalent to the fixed-point equation $S = \tilde{F}(S)$, where
\[\tilde{F}(S) = \hat{F}_{\beta,a,u_t+k \ast u_t}(x(w_t + k \ast w_t) + f - f, \tilde{u}_0 - u_t, \tilde{g} - g, \tilde{w}_T - w_T)\]
Let, as before, $\|S\| = \|z\|_2 + \|w\|_{2+1,1+\frac{1}{2}}$. Using \((77)\) we deduce for $S = (\beta, w)$ and $S^j = (\beta^j, w^j), j = 1, 2$, the relations
\[\|\tilde{F}(S)\| \leq \lambda \left(1 + \|k\| \|S\|^2 + D\right),\]
\[\|\tilde{F}(S^1) - \tilde{F}(S^2)\| \leq \lambda(1 + \|k\|) \frac{S^1 + S^2}{2} \|S^1 - S^2\|.
\]
Like in the proof of Theorem 5, these relations imply that $\tilde{F}$ is the contraction in the ball
\[\mathcal{B} = \left\{ S : S \in \mathcal{J}, \; \|S\| \leq \frac{1}{\lambda(1 + \|k\|)} \left(1 - \sqrt{1 - 2\lambda^2(1 + \|k\|)D}\right) \right\}.
\]
This implies (i).

It remains to prove (ii). Assume $p \in (1, \min\{\frac{1}{2}, \frac{1}{2\lambda}\})$. Theorem 1 immediately implies $u \in C^{2+1,1+\frac{1}{2}}(Q)$ for the solution of \((91)\) and Lemma 2 yields $u_t \in W_{r_1}^{2,1}(\Omega)$.
Define $v = u_t + k \ast u_t$. Then $v$ belongs to $C^{2+1,1+\frac{1}{2}}(Q) \cap W_{r_1}^{2,1}(Q)$ and solves the problem
\[\beta(v_t + k \ast v_t) = Av + f_t + k \ast f_t \quad \text{in } Q,
\]
\[v = \frac{1}{\beta}(Au_0 + f) \quad \text{in } \Omega \times \{0\}, \quad B_1 v = g_t + k \ast g_t \quad \text{in } S.
\]
(102)

Observing \((98), (99)\) and Theorem 2 we obtain
(103)
\[v \geq 0, \quad v(\cdot, T) > 0 \quad \text{in } \Omega \quad \text{in case I (II)}.
\]

Further, observing the assumptions of (ii), we see that Lemma 2 holds for the solution $v$ of the problem \((102)\), too. Consequently, $v_t \in W_{r_1}^{2,1}(Q) \Rightarrow u_t \in W_{r_1}^{2,1}(Q)$.

The function $\hat{u} = v_t + k \ast v_t - \theta_{\beta,t}v$ satisfies the following problem:
\[\beta(\hat{u}_t + k \ast \hat{u}_t) = A\hat{u}_t + \hat{f}_t \quad \text{in } Q,
\]
\[\hat{u} = \frac{1}{\beta} A \left(\frac{1}{\beta}(Au_0 + f)\right) + f_t - \theta_{\beta,t}(Au_0 + f) \quad \text{in } \Omega \times \{0\}, \quad B_1 \hat{u} = \hat{g}_t \quad \text{in } S.
\]

Observing the assumptions of (ii) and applying Theorem 2 we obtain $\hat{u} \geq 0, \hat{u}(x, T) > 0$ for $x \in \Omega$. These relations with \((103)\) imply the inequalities in \((93)\). If, additionally, $r_2 > 0$ in $I \times \{T\}$ in case I, then in view of \((103)\) we get the inequality $u_t(\cdot, T) > 0$ in $\overline{\Omega}$ both in cases I and II. This yields the assertion $u_t + k \ast u_t \geq \delta$ in $\overline{\Omega} \times (T - \delta, T)$ of \((94)\). Finally, in case \((100)\) the initial condition, the right-hand side and boundary condition of the problem of $u_t$
\[\beta(u_t + k \ast u_t) = Au_t + f_t - k(A(0)u_0 + f(\cdot, 0)) \quad \text{in } Q,
\]
\[u_t = \frac{1}{\beta}(Au_0 + f) \quad \text{in } \Omega \times \{0\}, \quad B_1 u_t = g_t \quad \text{in } S
\]
are zero for $t \in (0, \delta_0)$. This implies the assertion $u_t = 0$ in $\overline{\Omega} \times (0, \delta)$ of \((94)\).

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REFERENCES


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Research Article

Inverse Problems for a Parabolic Integrodifferential Equation in a Convolutional Weak Form

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We deduce formulas for the Fréchet derivatives of cost functionals of several inverse problems for a parabolic integrodifferential equation in a weak formulation. The method consists in the application of an integrated convolutional form of the weak problem and all computations are implemented in regular Sobolev spaces.

1. Introduction

Many methods to solve inverse problems (e.g., the Landweber iteration, conjugate gradient method) use the Fréchet derivatives of the cost functionals of these problems [1]. The explicit formula for the Fréchet derivative in terms of the variation of the unknowns of the inverse problem contains the solution of an adjoint problem.

The derivation of the explicit formula for such a Fréchet derivative includes testing the direct problem with the solution of the adjoint problem and vice versa: testing the adjoint problem with the solution of the direct problem. In the case of the parabolic weak problem, such a procedure is cumbersome, because of the asymmetry of the properties of the solution and the test function. In the classical formulation of the parabolic weak problem (see, e.g., [2] and also [19] below), the test function must have higher time regularity than the weak solution. This means that in case of non-smooth coefficients neither the solution of the direct problem nor the solution of the adjoint problem can be used as a test function.

Another formulation of the weak parabolic problem consists in reducing the problem to an abstract Cauchy problem over the time variable (see, e.g., [3]). In such a case, a partial integration over the time has to be implemented within singular distributions in the derivation procedure.

In this paper, we present a new method that enables the deduction of the formulas for the Fréchet derivatives for cost functionals of inverse problems related to weak solutions of parabolic problems. The method is based on an integrated convolutional form of the weak direct problem. The requirements to the test function are weaker than in the classical case and coincide with the properties of the solution of the direct problem. All computations in the deduction procedure can be implemented within usual regular Sobolev spaces.

More precisely, we will consider inverse problems related to a parabolic integrodifferential equation that occur in heat flow with memory [4–6]. This equation contains a time convolution. Therefore, the convolutional form of the weak problem is especially suitable. Supposedly, the proposed method can be generalised to parabolic systems, as well.

2. Formal Direct Problem: Notation

Let $\Omega$ be an $n$-dimensional domain, where $n \geq 1$, and $\Gamma$ be the boundary of $\Omega$. Let $\Gamma = \Gamma_1 \cup \Gamma_2$, where either $\Gamma_1$ or $\Gamma_2$ is allowed to be an empty set. In case $n \geq 2$, we assume that $\Gamma$ is sufficiently smooth, mean $\Gamma_1 \cap \Gamma_2 = 0$, and for any $j \in \{1, 2\}$ it holds either $\Gamma_j = \emptyset$ or mean $\Gamma_j > 0$. Denote

\[
\Omega_t = \Omega \times (0, t), \quad \Gamma_{1,t} = \Gamma_1 \times (0, t), \quad \Gamma_{2,t} = \Gamma_2 \times (0, t),
\] (1)
for \( t \geq 0 \). Consider the problem (direct problem) to find \( u(x, t) : \Omega_T \rightarrow \mathbb{R} \) such that

\begin{align*}
\dot{u}_t &= Au - m \cdot \nabla u + f + \nabla \cdot \phi \quad \text{in } \Omega_T, \quad (2) \\
u &= u_0 \quad \text{in } \Omega \times \{0\}, \quad (3) \\
\nabla \cdot \nu u + m \cdot \nabla u &= \delta u + h + \nabla \cdot \phi \quad \text{in } \Omega_T, \quad (4)
\end{align*}

where \( T > 0 \) is a fixed number,

\begin{align*}
A &= \sum_{i,j=1}^n \left( a_{ij} u_j \right)_{x_i}, \\
\nu &= \sum_{j=1}^n a_{ij} \nu_j |_{x=0}.
\end{align*}

(6)

\( \nu = (\nu_1, \ldots, \nu_n) \) - outer normal of \( \Gamma_T \).

\begin{equation}
\begin{aligned}
m \ast \omega(t) &= \int_0^t m(t - \tau) \omega(\tau) \, d\tau \\
\end{aligned}
\end{equation}

(7)

denotes the time convolution. In case \( \Gamma_1 = \emptyset \) (\( \Gamma_2 = \emptyset \)), the boundary condition (4) and (5) is dropped.

The problem (2)–(5) describes the heat flow in the body \( \Omega \) with the thermal memory. Concerning the physical background, we refer the reader to [4, 6, 7]. The solution \( u \) is the temperature of the body and \( m \) is the heat flux relaxation (or memory) kernel. The boundary condition (5) is of the third kind where the term \( -\nabla \cdot \nu u + m \cdot \nabla u \) equals the heat flux in the direction of the conormal vector.

Let us introduce some additional notations. Let \( t > 0 \). We use the Sobolev spaces

\begin{equation}
W^r_{2,2} (\Omega) = \left\{ v : \Omega \rightarrow \mathbb{R} : \|v\|_{W^r_{2,2} (\Omega)} := \left[ \sum_{l=0}^{r} \|D^a v\|_{L^2 (\Omega)}^2 \right]^{1/2} < \infty \right\}.
\end{equation}

(8)

Continuous functions from \([0, t]\) to \( X \) endowed with the usual maximum norm \( \|v\|_{C([0, t]; X)} := \max_{t \in [0, t]} \|v(t)\| \). Moreover, let

\begin{equation}
L^p ((0, t) \times X) := \left\{ v : (0, t) \rightarrow X : \|v\|_{L^p ((0, t); X)} := \left[ \int_0^t \|v(s)\|^p \, ds \right]^{1/p} < \infty \right\}
\end{equation}

for \( 1 < p < \infty \).

\begin{equation}
L^\infty ((0, t) \times X) := \left\{ v : (0, t) \rightarrow X : \|v\|_{L^\infty ((0, t); X)} := \text{ess sup}_{t \in (0, T)} \|v(t)\| < \infty \right\}.
\end{equation}

(9)

By means of these spaces, we define the following important functional spaces:

\begin{equation}
\mathcal{U} (\Omega) = C \left( [0, t] ; L^2 (\Omega) \right) \cap L^2 ((0, t); W^r_{2,2} (\Omega)),
\end{equation}

\begin{equation}
\mathcal{U}_a (\Omega) = \left\{ \eta \in \mathcal{U} (\Omega) : \eta |_{\Gamma_2} = 0 \right\}.
\end{equation}

(10)

Convention. In case \( n = 1 \), the integrals \( \int_{\Gamma_{j-1}} v(x) \, dx \), \( j = 1, \ldots, k \), are equal to \( \sum_{k=1}^K v(x_k) \), where \( x_k \in \Gamma_j \) and \( K \) is the number of points in \( \Gamma_j \) and \( L^p (\Gamma_j) \) is simply \( \mathbb{R}^K \).

3. Weak Direct Problem and Its Convolutional Form

Let us return to the direct problem (2)–(5). Throughout the paper we assume the following basic regularity conditions on the coefficients, the kernel, and the initial and boundary functions:

\begin{equation}
\begin{aligned}
a_{ij} &\in L^\infty (\Omega), & a_{ij} &= a_{ji}, & \theta \in C (\bar{\Omega}), & \theta \geq 0, \\
a &\in L^q (\Omega), & m &\in L^1 (0, T), & g &\in L^{r_1} (0, T; W^r_{2,2} (\Omega)), \quad (11) \\
g_l &\in L^{r_l} (\Omega), & f &\in L^2 ((0, T); L^b (\Omega)), & \phi &\in L^2 (\Omega), \quad (13)
\end{aligned}
\end{equation}

(12)

\begin{equation}
\begin{aligned}
m &\in L^1 (0, T), & g &\in L^1 (0, T; W^r_{2,2} (\Omega)), & g_l &\in L^{r_l} (\Omega), \\
f &\in L^2 ((0, T); L^b (\Omega)), & \phi &\in L^2 (\Omega), \quad (14)
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
u_0 &\in L^2 (\Omega), & h &\in L^2 (\Gamma_{2,T})
\end{aligned}
\end{equation}

(16)
and the ellipticity condition
\[ \sum_{i,j=1}^{n} a_{ij} \lambda_{i} \lambda_{j} \geq \epsilon | \lambda |^{2}, \quad x \in \Omega, \ \lambda \in \mathbb{R}^{n} \text{ with some } \epsilon > 0, \]
(17)
(for the sake of simplicity we introduced an assumption for the extension of \( g \) onto \( \Omega_{T} \)).

The first aim is to reformulate the problem (2)–(5) in a
weak form. Let us suppose that \( a_{ij} \in W^{1,1}_{c}(\Omega), (\partial/\partial x_{i}) \phi \in L^{2}(\Omega_{T}), i = 1, \ldots, n \) and (2)–(5) has a classical solution \( u \in W^{2,1}_{c}(\Omega_{T}). \) Then, we multiply (2) with a test function \( \eta \) from the space
\[ \mathcal{F} \left( \Omega_{T} \right) = \{ \eta \in L^{2}(0, T) ; W^{2,1}_{c}(\Omega) : \eta \in L^{2}(0, T) ; L^{2}(\Omega) \}, \]
and integrate by parts with respect to time and space variables.

We obtain the following relation:
\[ 0 = \int_{\Omega} (u(x, T) \eta(x, T) - u_{0}(x) \eta(x, 0)) \ dx - \int_{\Omega} u_{\eta} \ dx \ dt + \int_{\partial \Omega} \left( \sum_{i=1}^{n} a_{ij} (u_{x_{j}} - m \ast u_{x}) \eta_{x_{i}} - a (u - m \ast u) \eta \right) \ dx \ dt + \int_{\Omega_{T}} (9u + h) \eta \ dx \ dt - \int_{\Omega_{T}} (f - \phi \cdot \nabla \eta) \ dx \ dt. \]
(19)

This relation makes sense also in a more general case when \( a_{ij}, \phi \) satisfies only (11) and (15) and \( u \) does not have regular first-order time and second-order spatial derivatives.

We call a weak solution of the problem (2)–(5) a function from the space \( \mathcal{H}(\Omega_{T}) \) that satisfies the relation (19) for any \( \eta \in \mathcal{F}(\Omega_{T}) \) and in case \( \Gamma_{1} \neq \emptyset \) fulfills the boundary condition (4).

**Lemma 1.** The following assertions are valid.

(i) \( \mathcal{H}(\Omega_{T}) \rightarrow L^{2}(\Omega_{T}); L^{b}(\Omega) \) where \( q_{3} = \infty \) if \( n = 1, q_{3} \in (q_{1}, q_{2}/q_{1} - q_{3}), \infty \) if \( n = 2 \) and \( q_{3} = 2n/(n-2) \) if \( n > 2 \), where \( q_{1}, q_{2} \) are given in (12) and (14), respectively.

(ii) for any \( u \in L^{2}(\Omega_{T}); L^{b}(\Omega) \) it holds \( au \in L^{2}(\Omega_{T}); L^{b}(\Omega) \) and \( \| au \|_{L^{2}(\Omega_{T}); L^{b}(\Omega)} \leq C \| u \|_{L^{2}(\Omega_{T}); L^{b}(\Omega)} \), where \( C \) is a constant.

**Proof.** Since \( \mathcal{H}(\Omega_{T}) \rightarrow L^{2}(\Omega_{T}); W^{2,1}_{c}(\Omega) \), the assertion (i) follows from the continuous embedding of \( W^{2,1}_{c}(\Omega) \) in \( L^{b}(\Omega) \). The assertion (ii) can be proved by means of Hölder’s inequality. \( \square \)

**Theorem 2.** The problem (2)–(5) has a unique weak solution. This solution satisfies the estimate
\[ \| u \|_{\mathcal{H}(\Omega_{T})} \leq C_{0} \left( \| u_{0} \|_{L^{2}(\Omega)} + \| f \|_{L^{2}(\Omega_{T})} \right) + \| \phi \|_{L^{2}(\Omega_{T})}, \]
(20)
where \( \theta_{1} = 0 \) in case \( \Gamma_{1} = \emptyset, \theta_{2} = 0 \) in case \( \Gamma_{2} = \emptyset \) and \( C_{0} \) is a constant depending on \( \Omega, \Gamma_{1}, \Gamma_{2}, u, \theta \) and \( m \).

**Proof.** The assertion of the theorem in case \( m = 0 \) is well known from the theory of parabolic equations (see, e.g., [2]).

Let \( \mathcal{Z} \) be the operator that assigns to the data vector \( d := (u_{0}, f, \phi, g, h) \) the weak solution of the problem (2)–(5) in case \( m = 0 \). Then it holds
\[ \| \mathcal{Z} d \|_{\mathcal{H}(\Omega_{T})} \leq \text{RHS}, \]
(21)
where RHS is the right-hand side of (20).

Further, let us formulate the problem for the difference \( v = u - \mathcal{Z} d \). Introducing the linear operator \( \mathcal{A} \) by the formula
\[ \mathcal{A} \eta = \mathcal{Z} \left( 0, -am \ast \eta_{x}, -\sum_{j=1}^{n} a_{ij} m \ast u_{x_{j}}, 0, 0 \right), \]
(22)
the weak problem (2)–(5) for the solution \( u \in \mathcal{H}(\Omega_{T}) \) equivalent to the following operator equation for the quantity \( v \):
\[ v = \mathcal{A} v + \mathcal{Z} d. \]
(23)

We have to estimate \( \mathcal{A} \). For this purpose, we firstly prove the following auxiliary inequality:
\[ \| m \ast \eta \|_{L^{2}(\Omega_{T})} \leq \int_{0}^{t} \| m (t-r) \|_{L^{2}(\Omega_{T})} \ dx, \]
(24)
for any \( p \geq 1 \) and \( \eta \in L^{2}(\Omega_{T}); L^{p}(\Omega) \).

Denoting \( \mathcal{Y}(t) = \eta(t), \| \mathcal{Y}(t) \|_{L^{p}(\Omega)} = \| \mathcal{Y}(t) \|_{L^{p}(\Omega)} \), making use of the following property of the Bochner integral:
\[ \int_{0}^{t} \| w(s, r) \|_{L^{p}(\Omega)} \ dx < \int_{0}^{t} \| w(s, r) \|_{L^{p}(\Omega)} \ dx \]
for functions \( w \in L^{1}(\Omega_{T}); L^{p}(\Omega) \) and the Cauchy’s inequality, the relation
(24) can be deduced by means of the following computations:

$$
\|m \ast y\|_{L^2((0,T),L^2(\Omega))} \\
= \left[ \int_0^T \int_0^T m(r) \phi(s-r) \, dr \| \phi(s-r) \|^2 \, ds \right]^{1/2} \leq I, \quad \text{where}
$$

$$
I = \left[ \int_0^T \left( \int_0^T m(r) \| \phi(s-r) \| \, dr \right)^2 \, ds \right]^{1/2} \\
= \left[ \int_0^T m(r) \left( \int_0^T \| \phi(s-r) \| \, ds \right)^2 \, dr \right]^{1/2} \\
\leq \left[ \int_0^T m(r) \left( \int_0^T \| \phi(s-r) \|^2 \, ds \right)^{1/2} \, dr \right]^{1/2} \\
\times \left[ \int_0^T \left( \int_0^T m(r) \| \phi(s-r) \| \, ds \right)^2 \, dr \right]^{1/2} \\
\leq \left[ \int_0^T m(r) \left( \int_0^T \| \phi(s-r) \|^2 \, ds \right)^{1/2} \, dr \right]^{1/2} \\
\times \left[ \int_0^T \left( \int_0^T m(r) \| \phi(s-r) \| \, ds \right)^2 \, dr \right]^{1/4} \\
\leq \left[ \int_0^T m(r) \| \phi(s-r) \| \, ds \right]^{1/2} \\
\times \left[ \int_0^T m(r) \| \phi(s-r) \| \, dr \right]^{1/2} \\
= \left[ \int_0^T m(r) \| \phi(s-r) \| \, ds \right]^{1/2} \times T^{1/2}.
$$

Using Lemma 1, we obtain

$$
\|aw\|_{L^2((0,T),L^2(\Omega))} \leq C_1 \|a\|_{L^\infty(\Omega)} \|w\|_{L^2((0,T),W^2_0(\Omega))}.
$$

Using this relation in (27), we arrive at the following basic estimate for $a^w$:

$$
\|a^w\|_{W^1(\Omega_t)} \leq C_2 \int_0^T m(t-r) \|w\|_{W^1(\Omega_t)} \, dr, \quad t \in [0,T],
$$

where $C_2$ is a constant depending on $\Omega_\tau, \Gamma_\tau, a_j, \sigma, \theta$. Let us define the weighted norms in $W(\Omega_t)$: $\|v\|_{\sigma} = \sup_{\sigma \in C^0} e^{-\sigma t} \|v\|_{W(\Omega_t)}$ where $\sigma \geq 0$. The estimate (29) implies the further estimate

$$
\|a^w\|_{\sigma} \leq C_2 \sup_{\sigma \in C^0} e^{-\sigma t} \int_0^T m(t-r) \|w\|_{W(\Omega_t)} \, dr \\
= C_2 \sup_{\sigma \in C^0} \int_0^T e^{-\sigma(t-r)} m(t-r) \|w\|_{W(\Omega_t)} \, dr \\
\leq C_2 \int_0^T e^{-\sigma t} m(s) \, ds \|w\|_{\sigma}.
$$

Since $\int_0^T e^{-\sigma t} m(s) \, ds \to 0$ as $\sigma \to \infty$, there exists $\sigma_0$ depending on $C_2$ and $m$, such that $C_2 \int_0^T e^{-\sigma_0 t} m(s) \, ds \leq 1/2$. Thus, $\|a^w\|_{\sigma_0} \leq (1/2) \|w\|_{\sigma_0}$. The operator $a^w$ is a contraction in $W(\Omega_t)$. This implies the existence and uniqueness assertions of the theorem.

To prove the estimate (20), we firstly deduce from (23) the inequality $\|v\|_{\sigma_0} \leq \|a^w v\|_{\sigma_0} + \|\mathcal{Z} d\|_{\sigma_0} \leq (1/2) \|v\|_{\sigma_0} + \|\mathcal{Z} d\|_{\sigma_0}$. This implies $\|v\|_{\sigma_0} \leq \|\mathcal{Z} d\|_{\sigma_0}$. Using the equivalence relations $e^{-\sigma t} |\cdot|_{\sigma_0} \leq \|\cdot\|_{\sigma_0} \leq \|\cdot\|_{\sigma_0}$, we reach (20).

We note the upper integration bound $T$ in (19) can be released to be any number $t$ from the interval $[0,T]$. Indeed, (19) is equivalent to the following problem:

$$
0 = \int_{\Omega_t} \left[ \mathcal{L} u(x,t) \eta(x,t) - u_0(x) \eta(x,0) \right] \, dx - \int_{\Omega_t} u_\eta_0 \, dx \\
+ \int_{\Omega_t} \left[ \sum_{j=1}^n a_j \left( u_{x_j} - m \ast u_{x_j} \right) \right] \eta \, dxdt \\
- \int_{\Omega_t} \left[ \sum_{j=1}^n a_j (u - m) \ast u_{x_j} \right] \eta \, dxdt.
$$
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\[ + \int_{t_1}^{t_2} (\theta u + h) \eta dt \]
\[ - \int_{t_1}^{t_2} (f \eta - \phi \cdot \nabla \eta) dx dt, \quad t \in [0, T], \]  

(31)

for any \( \eta \in \mathcal{F}(\Omega_T) \). This assertion can be proved using the standard technique defining the test function as follows:

\[ \eta^\varepsilon(x, t) = \begin{cases} 
\eta(x, t) & \text{for } t \in [0, t_1], \\
\eta(x, t) \left(1 - \frac{t - t_1}{\varepsilon}\right) & \text{for } t \in (t_1, t_2), \\
0 & \text{for } t \in [t_2, T],
\end{cases} \]

(32)

and letting the parameter \( \varepsilon \) approach 0.

Next we transform the weak direct problem (31) to a form that does not contain a time derivative of the test function \( \eta \). This form enables the extension of the test space. This is useful for treatment of problems for adjoint states of quasisolutions of inverse problems in next sections.

**Theorem 3.** The function \( u \in \mathcal{U}(\Omega_T) \) satisfies the relation (19) for any \( \eta \in \mathcal{F}(\Omega_T) \) if and only if it satisfies the following relation:

\[ 0 = \int_{\Omega} u \ast \eta dx - \int_{\Omega} \int_{0}^{t} u_0(x) \eta(x, r) dr dx 
+ \int_{\Omega} 1 \ast \sum_{i,j=1}^{n} a_{ij} (u_{x_i} \ast m \ast u_{x_j}) \ast \eta_{x_i} 
- a(u \ast m \ast u) \ast \eta dx 
+ \int_{\Omega} 1 \ast (\theta u + h) \ast \eta dt 
- \int_{\Omega} 1 \ast \left( f \ast \eta - \sum_{i=1}^{n} \phi_i \ast \eta_{x_i} \right) dx, \quad t \in [0, T], \]

(33)

for any \( \eta \in \mathcal{U}_0(\Omega_T) \).

Here, according to the definition of the time convolution in the previous section, \( 1 \ast u(t) = \int_{0}^{t} u(\tau)d\tau \).

**Proof.** It is sufficient to prove that \( u \in \mathcal{U}(\Omega_T) \) satisfies (31) for any \( \eta \in \mathcal{F}(\Omega_T) \) if and only if it satisfies (33) for any \( \eta \in \mathcal{U}_0(\Omega_T) \). Suppose that \( u \in \mathcal{U}(\Omega_T) \) satisfies (31) and choose an arbitrary \( \eta \in \mathcal{F}(\Omega_T) \) such that the relation

\[ \xi^t(x, t, t) = \eta(x, t_1 - t) \quad \text{for } t \in [0, t_1] \]

(34)

is valid. For instance, it is possible to define \( \xi^t \) as a periodic function with respect to \( t \), that is, \( \xi^t(x, t) = \eta(x, t_1 - t) \) for

\[ t \in [0, t_1], \quad \xi^t(x, t) = \eta(x, t_1 - t_1) \quad \text{for } t \in [t_1, t_2], \quad \xi^t(x, t) = \eta(x, t_1 - t_1) \quad \text{for } t \in [t_2, t_3], \]

and so on. Using the relation (31) with \( \eta \) replaced by \( \xi^t \) and setting there \( t = t_1 \), we obtain the equality

\[ 0 = K_1(t_1) + K_2(t_1), \]

(35)

where

\[ K_1(t_1) = \int_{\Omega} [u(x, t) \eta(x, 0) - u_0(x) \eta(x, t)] dx 
+ \int_{\Omega} \int_{0}^{t} u(x, \tau) \eta_{x}(x, t - \tau) d\tau dx, \]

(36)

\[ K_2(t_1) = \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} (u_{x_i} - m \ast u_{x_j}) \ast \eta_{x_i} 
- a(u \ast m \ast u) \ast \eta dx 
+ \int_{\Omega} (\theta u + h) \ast \eta dt 
- \int_{\Omega} \left( f \ast \eta - \sum_{i=1}^{n} \phi_i \ast \eta_{x_i} \right) dx. \]

(37)

Note that the time derivative of \( \eta \) can be removed from \( K_1 \) by integration. Indeed, let \( t_2 \in [0, T] \). Then

\[ \int_{0}^{t_2} K_1(t_1) dt_1 = \int_{0}^{t_2} \int_{\Omega} u(x, t_1) \eta(x, 0) dx dt_1 
- \int_{0}^{t_2} \int_{\Omega} u_0(x) \eta(x, t_1) dx dt_1 \]

(38)

Changing the order of the integrals over \( r \) and \( t_1 \) in the last term, we easily obtain

\[ \int_{0}^{t_2} K_1(t_1) dt_1 = \int_{0}^{t_2} \int_{\Omega} u(x, \tau) \eta(x, t_2 - \tau) d\tau dx 
- \int_{0}^{t_2} \int_{\Omega} u_0(x) \eta(x, t_2) dx dt_1. \]

Integrating now the whole equality (35) over \( t_1 \) from 0 to \( t_2 \), observing (37) and (39), and finally redenoting \( t_2 \) by \( t \), we reach the desired relation (33). Summing up, we have proved that (33) holds for any \( \eta \in \mathcal{F}(\Omega_T) \). But all terms in the right-hand side of (33) are well defined for \( \eta \in \mathcal{U}_0(\Omega_T) \), too. Since \( \mathcal{F}(\Omega_T) \) is densely embedded in \( \mathcal{U}_0(\Omega_T) \), we conclude that (33) holds for any \( \eta \in \mathcal{U}_0(\Omega_T) \).

It remains to show that (33) implies (31). Suppose that \( u \in \mathcal{U}(\Omega_T) \) satisfies (33) and choose an arbitrary \( \eta \in \mathcal{F}(\Omega_T) \)
and $t_1 \in [0, T]$. Again, let $\xi^i$ be a function from $\mathcal{F}(\Omega_T)$ such (34) is valid. Inserting $\xi^i$ instead of $\eta$ into (33), differentiating with respect to $t$ and setting $t = t_1$ we come to the relation (31). Theorem is proved.

**Corollary 4.** A function $u \in \mathcal{H}(\Omega_T)$ is a weak solution of (2)–(5) if and only if it satisfies the relation (33) for any $\eta \in \mathcal{H}_0(\Omega_T)$ and in case $\Gamma_1 \neq \emptyset$ fulfills the boundary condition (4).

### 4. Inverse Problems and Quasisolutions

In the sequel, let us pose some inverse problems for the weak solution of (2)–(5). These problems are selected in order to demonstrate the wide possibilities of the method that we will introduce in Section 5.

Firstly, we suppose that (2)–(5) has the following specific form:

$$
\begin{align*}
\frac{\mathrm{d}u}{\mathrm{d}t} &= Au - m \star Au + f_0 + \nabla \cdot \phi + \sum_{j=1}^{N} \gamma_j(t) \omega_j(x) \quad \text{in } \Omega_T, \\
\frac{\partial u}{\partial n} &= g \quad \text{on } \Gamma_1, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \Gamma_2,
\end{align*}
$$

(40)

where $\omega = (\omega_1, \ldots, \omega_N)$ is unknown. The coefficients and other given functions $f_0, \gamma, \phi, g, h$ are assumed to satisfy (11)–(17). Moreover, $\gamma \in (L^2(0, T))^N$ is prescribed.

**IP1.** Find the vector $\omega \in (L^2(\Omega))^N$ such that the weak solution of (40) satisfies the following initial conditions:

$$
u(x, T_i) = \omega_{T_i}(x), \quad x \in \Omega, \quad i = 1, 2, \ldots, N,
$$

(41)

where $0 < T_1 < T_2 < \cdots < T_N \leq T$ and $\omega_{T_i} \in L^2(\Omega)$, $i = 1, 2, \ldots, N$ are given functions (observations of $u$).

Since $\sum_{j=1}^{N} \gamma_j \omega_j \in L^2((0, T); L^2(\Omega)) \subset L^2((0, T); L^b(\Omega))$ for $\omega \in (L^2(\Omega))^N$, the weak solution $u$ of (40) exists in $\mathcal{H}(\Omega_T)$; hence, it has traces $u(T_i) \in L^2(\Omega)$, $i = 1, 2, \ldots, N$. In practice, the term $\sum_{j=1}^{N} \gamma_j \omega_j$ may represent an approximation of a more general function $F(x, t)$ in $L^2(\Omega_T)$, where $F_j, j = 1, 2, \ldots$ form a basis in $L^2(0, T)$.

Further, let $u_{T_1}$ be unknown.

**IP2.** Find the vector $\omega \in (L^2(\Omega))^N$ and $u_0 \in L^2(\Omega)$ such that the weak solution of (40) satisfies the following integral additional conditions:

$$
\int_{T_0}^{T} \kappa(u(x, t)) \frac{\mathrm{d}x}{\mathrm{d}t} \, dt = v_j(x), \quad x \in \Omega, \quad i = 1, 2, \ldots, N + 1,
$$

(42)

where $v_i \in L^2(\Omega)$, $i = 1, \ldots, N + 1$ are given observation functions and $\kappa_1, i = 1, \ldots, N$ are given weights that satisfy the following condition:

$$
|\kappa_1(x, t)| \leq \kappa(t) \quad \text{in } \Omega_T, \quad i = 1, \ldots, N + 1
$$

(43)

with some $\kappa \in L^2(0, T)$.

Note that the integral $\int_{T_0}^{T} \kappa_1(x, t) \frac{\mathrm{d}x}{\mathrm{d}t} \, dt$ in (42) belongs to $L^2(\Omega)$ for any $\omega \in (L^2(\Omega))^N$ and $u_0 \in L^2(\Omega)$. Indeed, for such $\omega$ and $u_0$ it holds $u \in \mathcal{H}(\Omega_T) \subset L^2(\Omega_T)$, which implies

$$
\int_{T_0}^{T} \kappa_1(x, t) \frac{\mathrm{d}x}{\mathrm{d}t} \, dt \leq \kappa \in L^2(0, T)
$$

(44)

Note that $\|\kappa\|_{L^2(0, T)} \|\eta\|_{L^2(\Omega_T)} < \infty$.

In practice, the weights $\kappa_1$ are usually concentrated in neighborhoods of some fixed values of time $t = T_1$.

Finally, let us pose a nonlinear inverse problem for the coefficient $a$ and the kernel $m$. Assume that $m \in (1, 2, 3)$. Then any coefficient $a$ that belongs to $L^2(\Omega)$ satisfies (12). Moreover, let us set $q_1 = 2$ if $n = 2$ and $\Gamma_1 \neq \emptyset$. The other coefficients and the given functions $u_0, f, \phi, g, h$ are assumed to satisfy (11)–(17).

**IP3.** Find $a \in L^2(\Omega)$ and $m \in L^1(0, T)$ such that the weak solution of (2)–(5) satisfies the following integral additional conditions:

$$
u(x, T_i) = u_T(x), \quad x \in \Omega, \quad i = 1, 2, \ldots, N,
$$

(45)

where $u_T \in L^2(\Omega)$, $v \in L^2(0, T)$ are given observation functions and $\kappa$ is a given weight function such that $\kappa \in L^\infty((0, T); L^2(\Omega))$.

As in IP1, we can show that the trace $u(T, \cdot)$ belongs to $L^2(\Omega)$. Moreover, using the property $u \in \mathcal{H}(\Omega_T)$, the embedding of $W^2_0(\Omega)$ in $L^2(\Omega)$ and Hölder’s inequality, one can immediately check that the term $\int_{T_0}^{T} \kappa(x, \cdot) u(x, \cdot) \, dt$ in (45) belongs to $L^2(0, T)$.

Available existence, uniqueness, and stability results for IP1–IP3 require stronger smoothness of the data than imposed in the present paper. Let us cite some of these results.

In case $N = 1$, the well posedness of IP1 was proved in [8]. Partial results were deduced earlier in [9]. A more general problem involving both IP1 and IP2 without the unknown $u_0$ in case $N = 1$ was studied in [10] by means of different techniques. IP1 and IP2 in case $m = 0$ and $N = 1$ were treated in many papers, for example, [11–14]. The case $N > 1$ is open even if $m = 0$. Inverse problems to determine $m$ with given $a$ were studied in a number of papers, for example, [7, 15–23]. The problem for $a$ with given $m$ was treated in [8].

In the present paper, we will deal with quasisolutions of IP1–IP3 and related cost functionals. Denote $\mathcal{Z}_1 = (L^2(\Omega))^N$. 

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Let $M \subseteq Z_1$. The quasi-solution of IP1 in the set $M$ is an element $\omega^* \in \text{arg \, min}_{\omega \in M} J_1(\omega)$, where $J_1$ is the following cost functional

$$J_1(\omega) = \sum_{i=1}^{N} \left[ u(x, T; \omega) - u_{T}(x) \right]_{L^2(\Omega)}^2$$

(46)

and $u(x, t; \omega)$ is the solution of (40) that corresponds to a fixed element $\omega$.

Similarly, let $M \subseteq Z_2 := (L^2(\Omega))^{N+1}$. The quasi-solution of IP2 in the set $M$ is $z^* \in \text{arg \, min}_{z \in M} J_2(z)$, where $J_2$ is the cost functional

$$J_2(z) = \sum_{i=1}^{N+1} \int_{0}^{T} \kappa_i(x, t) u(x, t; z) \, dt - v_i(x) \left[ \int_{L^2(\Omega)}^2 \right.$$  

(47)

and $u(x, t; z)$ is the weak solution of (40) that corresponds to a given vector $z = (\omega, u_0)$.

Finally, defining $M \subseteq Z_3 := L^2(\Omega) \times L^2(0, T)$, the quasi-solution of IP3 in $M$ is an element $z^* \in \text{arg \, min}_{z \in M} J_3(z)$, where $J_3$ is the cost functional

$$J_3(z) = \left[ u(x, T; z) - u_{T}(x) \right]_{L^2(\Omega)}^2 + \left[ \int_{\Gamma_1} \kappa (x, t) u(x, t; z) \, d\Gamma - v (t) \right]_{L^2(0,T)}^2$$

(48)

and $u(x, t; z)$ is the weak solution of the direct problem (2)–(5) corresponding to given $z = (a, m)$. Here, we restricted the space for the unknown $m$ to $L^2(0, T)$, because we will seek for the Fréchet derivative of $J_3$ in a Hilbert space. Moreover, the kernel of the second addend corresponding to $m$ in the representation formula of $J_3$ (90) is an element of $L^2(0, T)$ and in general does not belong to the adjoint space $L^\infty(0, T)$.

According to the above-mentioned arguments, the functionals $J_k, k = 1, 2, 3,$ are well-defined in $Z_1$, $Z_2$, and $Z_3$, respectively.

5. The Fréchet Derivatives of Cost Functionals of Inverse Problems

5.1. General Procedure. Suppose that the solution $u$ of the direct problem depends on a vector of parameters $p$ that has to be determined in an inverse problem making use of certain measurements of $u$. Let the quasi-solution of the inverse problem be sought by a method involving the Fréchet derivative of the cost functional (i.e., some gradient-type method). Usually in practice, a solution of a proper adjoint problem is used to represent the Fréchet derivative.

We introduce a general procedure to deduce such adjoint problems. Assume that $\Delta u$ is the difference of solutions of the direct problem corresponding to a difference of the vector of the parameters $\Delta p$. More precisely, we suppose that $\Delta u$ is a solution of the following problem:

$$\Delta u = \Delta u - m \ast \Delta u + f^* + \nabla \cdot \Phi^* \quad \text{in} \quad \Omega_T,$$

(49)

$$\Delta u = \Delta u_0 \quad \text{in} \quad \Omega \times \{0\},$$

(50)

$$\Delta u = 0 \quad \text{in} \quad \Gamma_{1,T},$$

(51)

$$-\nu_A \cdot \nabla \Delta u + m \ast \nu_A \cdot \nabla \Delta u = 0 \quad \text{in} \quad \Gamma_{2,T},$$

(52)

with some data $f^*, \Phi^*, \Delta u_0, h^*$ depending on $\Delta p$. We restrict ourselves to the case when the Dirichlet boundary condition of the state $u$ is independent of $p$. Therefore, the condition (51) for $\Delta u$ is homogeneous.

In practice, the adjoint parabolic problems are usually formulated as backward problems. In our context, it is better to pose adjoint problems in the forward form. The involved memory term with $m$ is defined via a forward convolution and from the practical viewpoint, it is preferable to have the direct and adjoint problems in a similar form (e.g., to simplify parallelisation of computations).

More precisely, let the adjoint state $\psi$ be a solution of the following problem:

$$\psi_t = A\psi - m \ast \psi = f^* + \nabla \cdot \Phi^* \quad \text{in} \quad \Omega_T,$$

$$\psi = \Delta u \quad \text{in} \quad \Omega \times \{0\},$$

$$\psi = 0 \quad \text{in} \quad \Gamma_{1,T},$$

$$-\nu_A \cdot \nabla \psi + m \ast \nu_A \cdot \nabla \psi = 0 \quad \text{in} \quad \Gamma_{2,T},$$

(53)

where $f^*, \Phi^*, u^*$, and $h^*$ are some data depending on $\Delta u$ and the cost functional under consideration.

Assume that the quadruplets $f^*, \Phi^*, \Delta u_0, h^*$, and $f^*, \Phi^*, u^*, h^*$ satisfy the conditions (14)–(16). Then, due to Theorem 2, the problems (49)–(52) and (53) have unique weak solutions in the space $H(\Omega_T)$. Actually, we have $\Delta u, \psi \in H(\Omega_T)$ because of the homogeneous boundary conditions on $\Gamma_{1,T}$.

Let us write the relation (33) for $\Delta u$ and use the test function $\eta = \psi$. Then we obtain for any $t \in [0, T]$

$$0 = \int_{\Omega} \Delta u \ast \psi \, dx - \int_{\Gamma_1} \int_{0}^{t} \Delta u_0 (x) \, \psi (x, \tau) \, d\tau \, dx$$

$$+ \int_{\Omega} 1 \ast \left[ \sum_{i,j=1}^{n} a_{ij} \left( \Delta u_{x_i} - m \ast \Delta u_{x_j} \right) \cdot \psi_{x_i} \right.$$  

$$- a (\Delta u - m \ast \Delta u) \cdot \psi \eta \right] \, dx$$

(54)

$$+ \int_{\Gamma_1} 1 \ast \left( \hat{\Delta \eta} + h^* \right) \cdot \psi \, d\Gamma$$

$$- \int_{\Omega} 1 \ast \left( f^* \cdot \psi - \sum_{i=1}^{n} f_{x_i} \ast \psi_{x_i} \right) \, dx,$$
Secondly, let us write this relation for \( \psi \) and use the test function \( \eta = \Delta u \). Then we have for any \( t \in [0, T) \)
\[
0 = \int_{\Omega} \psi \ast \Delta u dx - \int_{\Omega} \int_{0}^{t} u'(x) \Delta u(x, t) \, dt \, dx \\
+ \int_{\Omega} 1 \left[ \sum_{i=1}^{n} \frac{a_{ij}}{a_{ii}} (\psi_{x_i} - m \ast \psi_{x_i}) \ast \Delta u_{x_i} \\
- a (\psi - m \ast \psi) \ast \Delta u \right] dx \\
+ \int_{\Gamma_{1,t}} 1 \left[ (\Psi \ast h') \ast \Delta u d\Gamma \right] \\
- \int_{\Omega} 1 \left[ f' \ast \Delta u - \sum_{i=1}^{n} \phi_{i} \ast \Delta u_{x_i} \right] dx.
\]
Subtracting (54) from (55), using the commutativity of the convolution, the symmetricity relations \( a_{ij} = a_{ji} \) and differentiating with respect to \( t \), we arrive at the following basic equality that can be used in various inverse problems:
\[
\int_{\Omega} u'(x) \Delta u(x, t) \, dx = \int_{\Omega} h' \ast \Delta u \, dx \\
+ \int_{\Omega} \left( f' \ast \Delta u - \sum_{i=1}^{n} \phi_{i} \ast \Delta u_{x_i} \right) dx \\
= \int_{\Omega} \Delta u_{0}(x) \psi(x, t) \, dx - \int_{\Gamma_{1,t}} h' \ast \psi d\Gamma \\
+ \int_{\Omega} \left( f' \ast \psi - \sum_{i=1}^{n} \phi_{i} \ast \psi_{x_i} \right) dx, \quad t \in [0, T).
\]
(56)

5.2. Derivative of \( J_{1} \)

\textbf{Theorem 5.} The functional \( J_{1} \) is the Fréchet differentiable in \( (L^{2}(\Omega))^{N} \) and
\[
J'_{1}(\omega) \omega = \sum_{j=1}^{N} \int_{\Omega} \int_{\Omega} \int_{0}^{T} \psi_{j}(x, t) \psi_{j} \, dt \, dx,
\]
(57)
where \( \psi_{j} \in \mathcal{V}(\Omega_{x}), i = 1, \ldots, N, \) are the unique \( \omega \)-dependent weak solutions of the following problems:
\[
\psi_{i} = A \psi_{i} - m \ast A \psi_{i} \quad \text{in} \quad \Omega_{x}, \\
\psi_{i} = 2 \left[ u(x, T_{i}; \omega) - u_{T_{i}}(x) \right] \quad \text{in} \quad \Omega \times (0), \\
\psi_{i} = 0 \quad \text{in} \quad \Gamma_{1,T_{i}}, \\
-\nu_{\lambda} \cdot \nabla \psi_{i} + m \ast \nu_{\lambda} \cdot \nabla \psi_{i} = \delta \psi_{i} \quad \text{in} \quad \Gamma_{2,T_{i}}, \\
i = 1, \ldots, N.
\]
(58)

\textbf{Proof.} Let us fix some \( \omega, \Delta \omega \in (L^{2}(\Omega))^{N} \). One can immediately check that it holds
\[
J_{1}(\omega + \Delta \omega) - J_{1}(\omega) \\
= 2 \sum_{i=1}^{N} \int_{\Omega} \left[ u(x, T_{i}; \omega) - u_{T_{i}}(x) \right] \Delta u(x, T_{i}; \omega) \, dx \\
+ \sum_{i=1}^{N} \int_{\Omega} \Delta u(x, T_{i}; \omega)^{2} \, dx,
\]
(59)
where \( \Delta u(x, t; \omega) = u(x, t; \omega + \Delta \omega) - u(x, t; \omega) \in \mathcal{V}_{0}(\Omega_{x,T}) \) is the weak solution of the following problem:
\[
\Delta u_{t} = A \Delta u - m \ast A \Delta u + \sum_{j=1}^{N} \gamma_{j} \Delta \omega_{j} \in \Omega_{T}, \\
\Delta u = 0 \quad \text{in} \quad \Omega \times (0), \\
\Delta u = 0 \quad \text{in} \quad \Gamma_{1,T_{i}}, \\
-\nu_{\lambda} \cdot \nabla \Delta u + m \ast \nu_{\lambda} \cdot \nabla \Delta u = \delta \Delta u \quad \text{in} \quad \Gamma_{2,T_{i}}.
\]
(60)

Applying the estimate (20) to the solution of this problem we deduce the following estimate for the second term in the right-hand side of (59):
\[
\sum_{i=1}^{N} \int_{\Omega} \Delta u(x, T_{i}; \omega)^{2} \, dx \leq n \|
\sum_{j=1}^{N} \gamma_{j} \Delta \omega_{j} \|^2_{(L^{2}(\Omega_{x,T}))^{n}} \leq C_{2} n \|
\sum_{j=1}^{N} \gamma_{j} \Delta \omega_{j} \|^2_{(L^{2}(\Omega))^{n}} \leq C_{4} \|
\Delta \omega \|^2_{(L^{2}(\Omega))^{n}},
\]
(61)
with some constant \( C_{4} \). This implies that \( J_{1} \) is the Fréchet differentiable and the first term in the right-hand side of (59) represents the Fréchet derivative, that is,
\[
J'_{1}(\omega) \Delta \omega = \sum_{i=1}^{N} \sigma_{i}
\]
with \( \sigma_{i} = 2 \int_{\Omega} \left[ u(x, T_{i}; \omega) - u_{T_{i}}(x) \right] \Delta u(x, T_{i}; \omega) \, dx.
\]
(62)

Further, let us use the method presented in Section 5.1 to deduce the proper adjoint problems. Comparing (60) with (49)–(52) we see that \( f' = \sum_{j=1}^{N} \gamma_{j} \Delta \omega_{j} \), \( \phi' = \Delta u_{0} = h' = 0 \). Therefore, the relation (56) has the form
\[
\int_{\Omega} u'(x) \Delta u(x, t) \, dx - \int_{\Gamma_{1,t}} h' \ast \Delta u d\Gamma \\
+ \int_{\Omega} \left( f' \ast \Delta u - \sum_{i=1}^{n} \phi_{i} \ast \Delta u_{x_i} \right) dx \\
= \sum_{j=1}^{N} \int_{\Omega} \gamma_{j} \Delta \omega_{j} \ast \psi d\Gamma, \quad t \in [0, T).
\]
(63)
In order to deduce a formula for the component \( s_j \) in the quantity \( J'_1(\omega) \Delta \omega \), we set \( u^i = u^i_0 = 2[u(x, T_i, \omega) - u^i_T(x)] \), \( h^i = f^i = 0 \) and \( t = T_i \) in (63). Then we immediately have

\[
\sigma_j = \sum_{j=1}^{N} \int_{\Omega} \gamma_j \Delta \omega \ast \psi_j dx \mid_{\omega=\omega_0},
\]

where according to (53) and the definition of \( u^i_0, h^i, f^i \), the function \( \psi_j \) is the weak solution of the problem (58) in the domains \( \Omega_{\tau_i} \) instead of \( \Omega_{\tau} \). Due to Theorem 2, this problem has a unique solution. From (62) and (64) we obtain (57). The latter formula contains the values of \( \psi_j \) in \( \Omega_{\tau_i} \). Therefore, we can restrict the problem (58) from \( \Omega_{\tau} \) to \( \Omega_{\tau_i} \).

To use the formula (57) one has to solve \( N \) weak problems for the functions \( \psi_j \) in domains \( \Omega_{\tau_i} \). In the following theorem, we will show that computational work related to the evaluation of the Fréchet derivative can be considerably reduced. Actually, it is sufficient to solve \( N \) weak problems in the smaller domains \( \Omega_{\tau_i-\tau_{i-1}}, i = 1, \ldots, N \). Here, \( \tau_0 = 0 \).

**Theorem 6.** The Fréchet derivative of the functional \( J_1 \) can also be written in the form

\[
J'_1(\omega) \Delta \omega = \sum_{j=1}^{N} \int_{\Omega} \sum_{l=1}^{l(N-1)} \int_{\Omega_{\tau_i-\tau_{i-1}}} \gamma_j (t) \beta_l (x, T_i) \Delta \omega \ast \psi_j dx dt,
\]

where \( \beta_l \in \mathcal{W}_0(\Omega_{\tau_i-\tau_{i-1}}) \) are the unique \( \omega \)-dependent weak solutions of the following sequence of recursive problems in the domains \( \Omega_{\tau_i-\tau_{i-1}} \):

\[
\begin{align*}
\beta_{l,i} &= \lambda \beta_l - m \ast \beta_l - a f^l - \nabla \cdot \Phi^l \quad \text{in} \quad \Omega_{\tau_i-\tau_{i-1}}, \\
\beta_l &= u^i_0 \quad \text{in} \quad \Omega \times [0, \tau_i], \\
\beta_l &= 0 \quad \text{in} \quad \Omega_{\tau_i,0},
\end{align*}
\]

\[
-\nabla \cdot \lambda \beta_l + m \ast \nabla \beta_l = 2 \beta_l \phi^l \ast \nabla \cdot \Phi^l \quad \text{in} \quad \Omega_{\tau_i-\tau_{i-1}},
\]

where \( l = N, N-1, \ldots, 1, \) Here,

\[
u^i_l (x) = x \left[ u (x, T_i, \omega) - u^i_T (x) \right] + \Theta_l \beta_{l+1} (x, T_{i+1} - T_i)
\]

and the function \( f^l \) and the vector \( \Phi^l \) are defined via \( \beta_{N-1}, \ldots, \beta_{l+1} \) as follows:

\[
f^l = \Theta_l \sum_{k=1}^{N-1} \int_{\Omega} m (T_k - T_i + t + \tau) \beta_{k+1} \times (x, T_{k+1} - T_k - \tau) dt,
\]

\[
\Phi^l = (\Phi^l_1, \ldots, \Phi^l_n), \quad \Phi^l_i = \sum_{j=1}^{n} a_{ij} (\partial / \partial x_j) f^l \quad \text{and} \quad \Theta_l = 0, \quad \theta_l = 1 \quad \text{for} \quad l < N.
\]

**Proof.** Firstly, let us check that (66) indeed have unique weak solutions \( \beta_l \in \mathcal{W}_0(\Omega_{\tau_i-\tau_{i-1}}) \). To this end we can use Theorem 2. For the problem \( \beta_{N,i} \) this is immediate, because the initial condition of the problem for \( \beta_{N} \) belongs to \( L^2(\Omega) \) and other equations in this problem are homogeneous. Further, we use the induction. Choose some \( l \in \text{range} \ N > l > 1 \) and suppose that \( \beta_{l+1} \in \mathcal{W}_0(\Omega_{\tau_{l+1}-\tau_l}) \) for all \( k \) such that \( N - 1 > k > l \). The aim is to use to show that then the problem for \( \beta_l \) has a unique weak solution in \( \mathcal{W}_0(\Omega_{\tau_i-\tau_{i-1}}) \).

Let us represent the \( k \)-th addend in (68) in the form

\[
I_k = \int_{0}^{T_{k+1}} \int_{0}^{T_{k+1} - T_k} m (T_k - T_i + t + \tau) \beta_{k+1} (x, T_{k+1} - T_k - \tau) d \tau dt = \int_{0}^{T_{k+1} - T_k} m (T_k - T_i + \tau) \beta_{k+1} (x, T_{k+1} - T_k + t - \tau) d \tau dt.
\]

For any \( k \) in the range \( N - 1 > k > 1 \) we have

\[
\left\| I_k \right\|_{L^2(\Omega_{\tau_{k+1}-\tau_k})} \leq \sum_{l=1}^{N} \int_{0}^{T_{k+1} - T_k} \int_{0}^{T_{k+1} - T_k} \int_{0}^{T_{k+1} - T_k} \left| m (T_k - T_i + t + \tau) \right| \left| \Theta_l \beta_{l+1} \right| d \tau dt dt,
\]

\[
\left\| D^{\ast} \beta_{k+1} : (T_{k+1} - T_k + t - \tau) \right\|_{L^1(\Omega)} dt \leq \sum_{l=1}^{N} \int_{0}^{T_{k+1} - T_k} \int_{0}^{T_{k+1} - T_k} \int_{0}^{T_{k+1} - T_k} \left| m (T_k - T_i + t + \tau) \right| z_{k,n} \left( T_{k+1} - T_k - \tau \right) d \tau dt dt,
\]

\[
\left\| \left( T_{k+1} - T_k - \tau \right) d \tau \right\|_{L^2(\Omega)} < \infty.
\]

This implies that \( f_l \) belongs to \( L^2(\Omega_{\tau_i-\tau_{i-1}}) \). From the latter relation and \( \theta_l \) we immediately have \( \Phi^l \in L^2(\Omega_{\tau_i-\tau_{i-1}}) \). Using the embedding theorem and Lemma 1 we see that \( \beta_{N,i} \) satisfies the property (14). Finally, the initial condition \( u^i_0 \) belongs to \( L^2(\Omega) \), because \( u - u^i_T \in C(\Omega, \tau_i - \tau_{i-1}, \Omega) \). All assumptions of Theorem 2 are satisfied for the problem for \( \beta_l \). Consequently, it possesses a unique weak solution in \( \mathcal{W}_0(\Omega_{\tau_i-\tau_{i-1}}) \).
Secondly, let us define the functions

$$
\beta_i^*(x, t) = \sum_{i=1}^{N} \psi_i(x, T_i - T_i + t) \quad \text{for (x, t) } \in \Omega_{i-1} - T_i - t.
$$

(71)

where \(i = 1, \ldots, N\) and \(\psi_i\) are the solutions of (58). We are going to show that \(\beta_i - \beta_i^*, i = 1, \ldots, N\). From the definition of \(\beta_i^*\) using the value of \(\psi_i(x, 0)\) and simple computations, we immediately get

$$
\beta_i^*(x, 0) = 2 \left[ u(x, T_i; \omega) - u_{T_i}(x) \right] + \Theta_i \beta_{i-1}^*(x, T_i - T_i).
$$

(72)

Let us fix \(i = 1, \ldots, N\) and choose some \(\eta \in \mathcal{F} \left( \Omega_{i-1} - T_i - t \right)\). We continue \(\eta\) by the formulae \(\eta(x, t) = \eta(x, T_i - T_i + t)\) for \(t > T_i - T_i - t\) and \(\eta(x, t) = \eta(x, 0)\) for \(t < 0\). Further, let us define \(\eta_i(x, t) = \eta(x, T_i - T_i + t)\) where \(i = 1, \ldots, N\). By the definition, it holds \(\eta_i \in \mathcal{F} \left( \Omega_i \right)\).

Let us write down the weak form (31) for the problem for \(\psi_i\) (58) with the test function \(\eta_i\). We fix some \(t \in [0, T_i - T_i - t]\) and compute the difference of this weak problem with \(t\) replaced by \(T_i - T_i + t\) and \(t\) replaced by \(T_i - T_i\) and take the sum over \(i = 1, \ldots, N\). This results in the following expression:

$$
0 = Z_1 + Z_2 + Z_3 + Z_4.
$$

(73)

Using the definitions of \(\eta\) and \(\beta_i^*\) and the formula (72), we have

$$
Z_1 = \int_{\Omega} \left[ \beta_i^*(x, t) \eta(x, t) - \beta_i^*(x, 0) \eta(x, 0) \right] dx
$$

$$
= \int_{\Omega} \left[ \beta_i^*(x, t) \eta(x, t) - \left\{ 2 \left[ u(x, T_i; \omega) - u_{T_i}(x) \right] + \Theta_i \beta_{i-1}^*(x, T_i - T_i) \right\} \eta(x, 0) \right] dx.
$$

(75)

Similarly, using the definitions of \(\eta\) and \(\beta_i^*\) and changing the variable of integration in \(Z_2\), we deduce

$$
Z_2 = \int_{\Omega} \left[ -\beta_i^* \eta_t + \sum_{i=1}^{n} a_{ij} \beta_i^* \eta_{x_i} - a \beta_i^* \eta \right] dx dt
$$

$$
+ \int_{\mathcal{F}} \left[ 8 \beta_i^* \eta \text{d}t \right].
$$

(76)

By the change of variable, the quantity \(Z_3\) is transformed to

$$
Z_3 = \int_{\Omega} a(x) \sum_{i=1}^{N} \left( m \psi_i(x, T_i - T_i + t) \right) \eta(x, t) dx dt.
$$

(77)

Let us consider the term \(\sum_{i=1}^{N} (m \psi_i(x, T_i - T_i + t))\) in the latter formula. We compute

$$
\sum_{i=1}^{N} (m \psi_i(x, T_i - T_i + t))
$$

$$
= \sum_{i=1}^{N} m(T_i - T_i + t) \psi_i(x, T_i - T_i + t) \left( \eta_t - \eta \right) dx dt
$$

$$
= \int_{\Omega} \sum_{i=1}^{N} m(T_i - T_i + t) \psi_i(x, T_i - T_i + t) \left( \eta_t - \eta \right) dx dt
$$

$$
= \left( m \psi_i(x, T_i - T_i + t) \right) (x, t)
$$

$$
+ \sum_{i=1}^{N} \int_{T_i}^{T_i + t} \psi_i(x, T_i - T_i + t) \left( \eta_t - \eta \right) \left( \eta_t - \eta \right) dx dt.
$$

$$
= \left( m \psi_i(x, T_i - T_i + t) \right) (x, t)
$$

$$
+ \sum_{i=1}^{N} \int_{T_i}^{T_i + t} \psi_i(x, T_i - T_i + t) \left( \eta_{T_i} - \eta \right) dx dt.
$$

(74)
\[ = (m \ast \beta^*_{i,j}) (x,t) \\
+ \sum_{k=1}^{N+1} \int_{T_{k-1}}^{T_k} m(T_k - T_i + t + \tau) \times \beta^*_{k+1} (x, T_{k+1} - T_k - \tau) \, dt. \]

Thus, (77) reads

\[ Z_3 = \int_{\Omega_i} m \ast \beta^*_{i,j} \eta dx dt \\
+ \int_{\Omega_i} \left[ \sum_{k=1}^{N+1} \int_{T_{k-1}}^{T_k} m(T_k - T_i + t + \tau) \times \beta^*_{k+1} (x, T_{k+1} - T_k - \tau) \, dt \right] \eta dx dt. \]

Using similar computations, we obtain

\[ Z_4 = -\int_{\Omega_i} \sum_{j=1}^{n} \sum_{i,j=1}^{n} a_{i,j} m \ast \beta^*_{i,j} \eta_i dx dt - \int_{\Omega_i} \sum_{i,j=1}^{n} a_{i,j} \times \left[ \sum_{k=1}^{N+1} \int_{T_{k-1}}^{T_k} m(T_k - T_i + t + \tau) \times \beta^*_{k+1,i,j} (x, T_{k+1} - T_k - \tau) \, dt \right] \eta_i dx dt. \]

Plugging (75), (76), (79), and (80) into (73), we arrive at a certain weak problem for \( \beta^*_{i,j} \) that coincides with the weak problem for \( \beta_i \). Moreover, since \( \psi_i \in \mathcal{W}_0(\Omega_{T_i}) \), from (71) we see that \( \beta^*_{i,j} \in \mathcal{W}_0(\Omega_{T_i-T_{i-1}}) \). But we have shown the uniqueness of the weak solutions of the problems for \( \beta_i \) in \( \mathcal{W}_0(\Omega_{T_i-T_{i-1}}) \). This implies \( \beta^*_{i,j} = \beta_i \).

Finally, from (57), we have

\[ J'_1 (\omega) \Delta \omega = \sum_{j=1}^{N} \int_{\Omega} \int_{T_i}^{T_{j+1}} \psi_i (x, T_i + t - \tau) \, dt \Delta \omega_j (x) \, dx \\
= \sum_{j=1}^{N} \int_{\Omega} \int_{T_i}^{T_{j+1}} \psi_i (x, T_i + t - \tau) \, dt \Delta \omega_j (x) \, dx. \]

Thus, (81) reads

\[ 5.3. \text{Derivative of } J_2 \]

Theorem 7. The functional \( J_2 \) is the Fréchet differentiable in \( (l^2(\Omega))^{N+1} \) and

\[ J'_2 (\omega) \Delta \omega = \sum_{j=1}^{N} \int_{\Omega} \int_{T_i}^{T_{j+1}} \psi_i (x, T_i + t - \tau) \, dt \Delta \omega_j (x) \, dx \\
+ \int_{\Omega} \psi(x, T_i + \tau) \Delta u_0 (x) \, dx, \]

where \( \psi \in \mathcal{W}(\Omega_T) \) is the unique \( z \)-dependent weak solution of the following problem:

\[ \psi_i = A \psi - m \ast A \psi \\
+ \sum_{i=1}^{N+1} \kappa_i (x, T_i - t) \times \left[ \int_{T_i}^{T} \kappa_i (x, \tau) u(x, \tau + \tau) \, d\tau - \psi_i (x) \right] \quad \text{in } \Omega_T, \]

\[ \psi = 0 \quad \text{in } \Omega \times \{0\}, \]

\[ \psi = 0 \quad \text{in } \Omega_{T_i-T_{i-1}}, \]

\[ -\nabla \cdot \nabla \psi + m * \nabla \psi = \nabla \psi \quad \text{in } \Omega_{ \sqrt{2} T_i} . \]

Proof. Let us fix some \( \Delta \omega = (\Delta \omega, \Delta u_0) \in (l^2(\Omega))^{N+1} \). It holds

\[ I_2 (\omega + \Delta \omega) - I_2 (\omega) \\
= 2 \sum_{i=1}^{N+1} \int_{\Omega} \int_{T_i}^{T} \kappa_i (x, t) \times \left[ \int_{T_i}^{T} \kappa_i (x, \tau) u(x, \tau + \tau) \, d\tau - \psi_i (x) \right] \quad \text{in } \Omega_T, \]

\[ + \sum_{i=1}^{N+1} \int_{\Omega} \left[ \int_{T_i}^{T} \kappa_i (x, \tau) \Delta u(x, \tau + \tau) \, d\tau \right] \, dx, \]

where \( \Delta u(x, t; z) = u(x, t; z + \Delta z) - u(x, t; z) \in \mathcal{W}(\Omega_T) \) is the weak solution of the following problem:

\[ \Delta u = A \Delta u - m * A \Delta u + \sum_{j=1}^{N} \psi_j \Delta \omega_j \quad \text{in } \Omega_T, \]

\[ \Delta u = \Delta u_0 \quad \text{in } \Omega \times \{0\}, \]

\[ \Delta u = 0 \quad \text{in } \Omega_{ \sqrt{2} T_i}, \]

\[ -\nabla \cdot \nabla \Delta u + m * \nabla \Delta u = \Delta \psi \quad \text{in } \Omega_{ \sqrt{2} T_i}. \]
Using (43), the Cauchy inequality and estimate (20) from Theorem 2 for the problem of $\Delta u(x, t; z)$, we come to the estimate

$$
\left| \sum_{i=1}^{N+1} \int_{\Omega} \left[ \int_{0}^{T} \kappa_i(x, t) \Delta u(x, t; z) \, dt \right] \, dx \right|
\leq (N + 1) \| \kappa \|_{L^2(\Omega, T)} \| \Delta u \|_{L^2(\Omega, T)}^2
\leq C_\kappa (N + 1) \| \kappa \|_{L^2(\Omega, T)} \| \Delta u \|_{W^2(\Omega)}^2 \leq C_\kappa \| \Delta u \|_{L^2(\Omega)}^{N+1},
$$

(86)

with some constants $C_\kappa$ and $C_\Delta$. Therefore, $J_2$ is the Fréchet differentiable and the first term in the right-hand side of (84) represents the Fréchet derivative, that is,

$$
J'_2(x) \Delta x = 2 \sum_{i=1}^{N+1} \int_{\Omega} \int_{0}^{T} \kappa_i(x, t) \nabla u(x, t; z) \, dt \, dx
\leq \sum_{i=1}^{N+1} \int_{\Omega} \kappa_i(x, t) \nabla u(x, t; z) \, dt \, dx
\leq \Delta u(x, t; z) \, dt \, dx.
$$

(87)

Comparing (85) with (49)–(52), we see that $f^T = \sum_{i,j=1}^{N} \gamma_{i,j} \Delta \omega_{i,j}$, $\phi^T = h^T = 0$. Consequently, the relation (56) has the form

$$
\int_{\Omega} u^T(x) \Delta u(x, t) \, dx - \int_{\Gamma_1} h^\ast \ast \Delta u \, d\Gamma
+ \int_{\Omega} \left( f^T \ast \Delta u - \sum_{i=1}^{N} \phi^T \ast \Delta u_{\kappa_i} \right) \, dx
= \int_{\Omega} \Delta u_0(x) \psi(x, t) \, dx
+ \sum_{j=1}^{N} \int_{\Omega} \gamma_{i,j} \Delta \omega_{i,j} \ast \psi \, dx \quad t \in [0, T].
$$

(88)

To deduce a formula for $J'_2(x) \Delta x$, we define

$$
\begin{aligned}
\Delta u_0 &= A \Delta u - m \ast A \Delta u + \Delta [u - m \ast u] - \Delta m \ast u \\
&\quad - \nabla \cdot \left[ \Delta m \ast \sum_{j=1}^{n} a_{ij} u_{r_j} \right] \quad \text{in } \Omega_T, \\
\Delta u &= 0 \quad \text{in } \Omega \times [0, T], \\
- \nabla \cdot \Delta u + m \ast \nabla \Delta u &= \Delta m \ast u - \Delta m \ast u
- \nabla \cdot \left[ \Delta m \ast \sum_{j=1}^{n} a_{ij} u_{r_j} \right] \quad \text{in } \Gamma_{1,T},
\end{aligned}
$$

(92)

Proof. Due to $u(x, t; z) \in \mathcal{W}(\Omega_T), \kappa \in L^\infty((0, T); L^2(\Gamma_1)), \nu \in L^2(0, T)$, and $u_{r} \in L^2(\Omega)$, the problem (91) satisfies the assumptions of Theorem 2. Therefore, it has a unique weak solution in $\mathcal{W}_0(\Omega_T)$.

Let $\Delta z = (\Delta a, \Delta m) \in L^2(\Omega) \times L^2(0, T)$ and define $\tilde{\Delta} u = u(x, t; z + \Delta z) - u(x, t; z)$. We split $\Delta u$ as follows: $\Delta u = \Delta u + \tilde{\Delta} u$, where $\Delta u$ is the weak solution of the following problem:

$$
\begin{align*}
\Delta u &= A \Delta u - m \ast A \Delta u + \Delta [u - m \ast u] - \Delta m \ast u \\
&\quad - \nabla \cdot \left[ \Delta m \ast \sum_{j=1}^{n} a_{ij} u_{r_j} \right] \quad \text{in } \Omega_T, \\
\Delta u &= 0 \quad \text{in } \Omega \times [0, T], \\
- \nabla \cdot \Delta u + m \ast \nabla \Delta u &= \Delta m \ast u - \Delta m \ast u
- \nabla \cdot \left[ \Delta m \ast \sum_{j=1}^{n} a_{ij} u_{r_j} \right] \quad \text{in } \Gamma_{1,T}.
\end{align*}
$$

(92)

In view of Lemma 1(i), $u \in \mathcal{W}(\Omega_T)$, $m \in L^1(0, T)$, and the Young’s theorem, it holds $u - m \ast u \in L^2((0, T); L^0(\Omega))$. 

5.4. Derivative of $J_3$

Theorem 8. The functional $J_3$ is the Fréchet differentiable in $L^2(\Omega) \times L^2(0, T)$ and

$$
J'_3(x) \Delta x = \int_{\Omega} \left[ \left( u - m \ast u \right) \ast \psi \right](x, T) \, \Delta u(x) \, dx
+ \int_{\Omega} \int_{0}^{T} \left[ \sum_{i,j=1}^{n} a_{ij} u_{r_j} \ast \psi \ast \Delta u - au \ast \psi \right] \, dx \, dt,
$$

(90)

where $\psi \in \mathcal{W}_0(\Omega_T)$ is the unique $z$-dependent weak solution of the problem

$$
\begin{align*}
\psi_t &= A \psi - m \ast A \psi \quad \text{in } \Omega_T, \\
\psi &= 2 \left[ u(x, T; z) - u_T(x) \right] \quad \text{in } \Omega \times (0, T), \\
\psi &= 0 \quad \text{in } \Gamma_{1,T}, \\
- \nu \cdot \nabla \psi + m \ast \nu \cdot \nabla \psi &= \delta \psi, \\
-2 \nu \cdot (x, T - t) \\
&\quad \times \left[ \int_{\Gamma_1} \kappa(y, T - t) u(y, T - t; z) \, d\Gamma - v(T - t) \right] \quad \text{in } \Gamma_{1,T}.
\end{align*}
$$

(91)
Therefore, Lemma 1(ii) implies
\[ \| \Delta u - m \cdot u \|_{L^2(0,T;L^\infty(\Omega))} \leq C_6 \| u, m \| \Delta u \|_{L^2(0,T)} \leq C_6 \| u, m \| \Delta u \|_{L^2(0,T)}, \] (93)
where \( C_6 \) and \( C_7 \) are some constants depending on \( u, m \).
Moreover, since \( u \in L^2((0,T);W^1_0(\Omega)) \), by Young's inequality we have also
\[ \left\| \Delta u + \sum_{j=1}^n a_j u_{x_j} \right\|_{L^2(0,T)} \leq C_{10} \| u, m \| \Delta u \|_{L^2(0,T)} \leq C_{11} \| u, m \| \Delta u \|_{L^2(0,T)}, \] (94)
with some constants \( C_{10} \) and \( C_{11} \) depending on \( u \). The obtained estimates show that assumptions of Theorem 2 are satisfied for the problem (92) and it indeed has a unique weak solution \( \Delta u \in \mathcal{H}(\Omega_T) \). Moreover, applying the relation (20) from Theorem 2, we get
\[ \| \Delta u \|_{\mathcal{H}(\Omega_T)} \leq C_{12} \| u, m \| \Delta u \|_{L^2(0,T)} = C_{12} \| u, m \| \Delta u \|_{L^2(0,T)}, \] (95)
where \( C_{12} \) is a constant depending on \( m, u \).
Further, writing the problem for \( \Delta u \) and subtracting the problem for \( \Delta u \), we obtain the following problem for \( \Delta u \):
\[ \Delta u = \Delta a \Delta u - m - \Delta a \Delta u + f + \bar{f} + \nabla \cdot \phi + \nabla \cdot \bar{\phi} \quad \text{in} \quad \Omega_T, \]
\[ \Delta u = 0 \quad \text{in} \quad \Omega \times \{0\}, \]
\[ \Delta u = 0 \quad \text{in} \quad \Gamma_{1,T}, \]
\[ -
abla \cdot \Delta u + m \cdot \nabla \cdot \Delta u = 8 \Delta u + \nabla \cdot \phi + \nabla \cdot \bar{\phi} \quad \text{in} \quad \Gamma_{3,T}, \] (96)
where
\[ f = \Delta a \Delta u - (m + \Delta m) \cdot \Delta a \Delta u - \Delta m \cdot \Delta a u - \Delta m \cdot a \Delta u, \]
\[ \bar{f} = \Delta a \Delta u - (m + \Delta m) \cdot \Delta a \Delta u - \Delta m \cdot a \Delta u, \]
\[ \phi = -
abla m \cdot \sum_{j=1}^n a_j \Delta u_{x_j}, \quad \bar{\phi} = -
abla m \cdot \sum_{j=1}^n a_j \Delta u_{x_j} \] (97)
Using again Lemma 1 and the Young's inequality, we deduce the estimates
\[ \| f \|_{L^2(0,T;L^\infty(\Omega))} \leq C_{13} \left\{ \left\| \Delta a \right\|_{L^2(\Omega)} + \left\| \Delta m \right\|_{L^2(\Omega)} + \left\| m \right\|_{L^2(\Omega)} \right\} \| \Delta u \|_{L^2(0,T)} \]
\[ + \left\| \Delta m \right\|_{L^2(\Omega)} \| a \|_{L^2(\Omega)} \| \Delta u \|_{W(\Omega_T)} \]
\[ + \left\| \Delta a \right\|_{L^2(\Omega)} \left\| \Delta m \right\|_{L^2(\Omega)} \| m \|_{L^2(\Omega_T)} \]
\[ \leq C_{14} (\varepsilon, u) \left\{ \left\| \Delta u \right\|_{W(\Omega_T)} + \left\| m \right\|_{L^2(\Omega)} \right\} \| \Delta u \|_{W(\Omega_T)} + \| \Delta u \|_{W(\Omega_T)}^2, \]
\[ \| f \|_{L^2(0,T;L^\infty(\Omega))} \leq C_{15} (\varepsilon, u) \left\{ \left\| \Delta u \right\|_{W(\Omega_T)} + \left\| m \right\|_{L^2(\Omega)} \right\} \| \Delta u \|_{W(\Omega_T)} + \| \Delta u \|_{W(\Omega_T)}^2, \]
\[ \| \phi \|_{L^2(\Omega_T)} \leq C_{16} \| \Delta m \|_{L^2(\Omega_T)} \| \Delta u \|_{W(\Omega_T)} \]
\[ \leq C_{16} \| \Delta u \|_{W(\Omega_T)} \| \Delta u \|_{W(\Omega_T)} \]
\[ \| \phi \|_{L^2(\Omega_T)} \leq C_{17} \| \Delta u \|_{W(\Omega_T)} \] (98)
with some constants \( C_{12}, \ldots, C_{17} \). Therefore, applying the relation (20) to the solution of the problem (96) we obtain
\[ \| \Delta u \|_{W(\Omega_T)} \leq C_{18} (\varepsilon, u) \]
\[ \times \left\{ \left\| \Delta u \right\|_{W(\Omega_T)} + \left\| m \right\|_{L^2(\Omega)} \right\} \| \Delta u \|_{W(\Omega_T)} + \| \Delta u \|_{W(\Omega_T)}^2 \}, \] (99)
with some constant \( C_{18} \). In case \( \| \Delta u \| \) is small enough, that is,
\[ \| \Delta u \| + \| \Delta u \| \leq \frac{1}{2 C_{18} (\varepsilon, u)}, \] (100)
we have
\[ \| \Delta u \|_{W(\Omega_T)} \leq C_{19} (\varepsilon, u) \]
\[ \times \left\{ \left\| \Delta u \right\|_{W(\Omega_T)} + \left\| m \right\|_{L^2(\Omega)} \right\} \| \Delta u \|_{W(\Omega_T)} + \| \Delta u \|_{W(\Omega_T)}^2 \}. \] (101)
In view of (95), this implies
\[ \| \Delta u \|_{W(\Omega_T)} \leq C_{19} (\varepsilon, u) \left\{ \left\| \Delta u \right\|_{W(\Omega_T)} + \left\| m \right\|_{L^2(\Omega)} \right\} \] (102)
with a constant \( C_{19} \).
Similarly, for the solution of the problem (92), we deduce the estimate
\[ \| \Delta u \|_{W(\Omega_T)} \leq C_{20} (\varepsilon, u) \| \Delta u \|_{W(\Omega_T)} \] (103)
with a constant \( C_{20} \).
Now, we write the difference of $J_3$ in the following form:

$$
J_3(x + \Delta x) - J_3(x) = \frac{1}{2} \int_\Omega \left[ u(x, T; z) - u_T(x) \right] \Delta u(x, T; z) \, dx + 2 \int_{0}^{T} \int_\Gamma \kappa(x, t) \left[ \int_\Gamma \kappa(y, t) u(y, t; z) \, d\Gamma - u(t) \right] \Delta u(x, t; z) \, d\Gamma \, dt + \Theta,
$$

where

$$
\Theta = 2 \int_\Omega \left[ u(x, T; z) - u_T(x) \right] \Delta u(x, T; z) \, dx + 2 \int_{0}^{T} \int_\Gamma \kappa(x, t) \left[ \int_\Gamma \kappa(y, t) u(y, t; z) \, d\Gamma - u(t) \right] \Delta u(x, t; z) \, d\Gamma \, dt + \int_\Omega \left( \left( \Delta u + \Delta u_T \right)(x, T; z) \right)^2 \, dx + \int_{0}^{T} \left( \int_\Gamma \kappa(x, t) \left( \Delta u + \Delta u_T \right)(x, t; z) \, d\Gamma \right)^2 \, dt.
$$

Using (102), (103), and the property $\kappa \in L^\infty((0, T); L^2(\Gamma_2))$, we obtain the estimate $|\Theta| \leq C_{2,1}(z, u) \sum_{i=1}^{k} |\Delta z_i|^2$ in case (100). This shows that $J_3$ is Fréchet differentiable and

$$
J'_3(\Delta x) = \frac{1}{2} \int_\Omega \left[ u(x, T; z) - u_T(x) \right] \Delta u(x, T; z) \, dx + 2 \int_{0}^{T} \int_\Gamma \kappa(x, t) \left[ \int_\Gamma \kappa(y, t) u(y, t; z) \, d\Gamma - u(t) \right] \Delta u(x, t; z) \, d\Gamma \, dt.
$$

In order to obtain a formula for the right-hand side in (106), we set $u' = 2[u(x, T; z) - u_T(x)]$,

$$
h'(x, t) = -2\kappa(x, T; t),
$$

and

$$
f' = \phi = 0 \text{ and } t = T. \text{ Then, we obtain (90), where in view of (53) the function } \psi \text{ is the weak solution of (91).} \quad \square
$$

6. Further Aspects of Minimisation

6.1. Existence of Quasisolutions. For the convenience, we will use also the symbol $x$ to denote the argument $z$ of $J_k$.

**Theorem 9.** (i) Let $k \in \{1, 2\}$ and $M \subset X_k$ be bounded, closed, and convex. Then, $IPk$ has a quasisolution in $M$. The set of quasisolutions is closed and convex.

(ii) Let $k \in \{1, 2, 3\}$ and $M \subset X_k$ be compact. Then $IPk$ has a quasisolution in $M$.

**Proof.** Let us prove (i). The existence assertion follows from Weierstrass existence theorem (see [24, Section 2.5, Theorem 2D]) once we have proved that $J_k$ is weakly sequentially lower semicontinuous in $X$, that is,

$$
J_k(x) \leq \liminf_{n \to \infty} J_k(x_n) \quad \text{as } x_n \rightharpoonup x \text{ in } X_k.
$$

But (109) follows from the continuity and convexity of $J_k$ [24]. The convexity of $J_k$ can be immediately deduced making use of the linearity of the ingredient $u(x, t; z)$ with respect to $z$ inside the quadratic functional $J_k$ (for similar computations see [25, Theorem 2]). The closedness of the set of quasisolutions is again a direct consequence of the continuity of $J_k$. The convexity of the set of solutions follows from the convexity of $J_k$.

Next, we prove (ii). Let $m = inf_{z \in X} J_k(z)$ and $z_i \in M$ be the minimising sequence, that is, $\lim J_k(z_i) = m$. By the compactness, there exists a subsequence $z_{i_j} \in M$ such that $\lim z_{i_j} = z^* \in M$. Due to the continuity of $J_k$ we have $\lim J_k(z_{i_j}) = J_k(z^*)$. Thus, $J_k(z^*) = m$. This proves (ii). \quad \square

Finally, let us prove (90) and (91). Comparing (92) with (49)–(52), we see that

$$
f^* = \Delta u - \sum_{i=1}^{n} \phi_i \Delta u_{x_i}, \quad \Delta u_0 = h^* = 0. \text{ Thus, (56) reads}
$$

$$
\begin{align*}
\int_{\Omega} u(x) \Delta u(x, t) \, dx & - \int_{\Gamma_1} h^* \Delta u \, d\Gamma \\
& + \int_{\Omega} \left( f^* - \Delta u - \sum_{i=1}^{n} \phi_i \Delta u_{x_i} \right) \, dx \\
= & \int_{\Omega} \left( \Delta u [-m - u] - \Delta m \cdot au \right) \psi \, dx \\
& + \Delta m \cdot \sum_{i,j=1}^{n} a_{ij} u_{x_i} \cdot \psi_{x_j} \, dx, \quad t \in [0, T].
\end{align*}
$$

In practice, the compact set $M$ may be a bounded and closed finite-dimensional subset of $X_k$. The proof of weak lower semicontinuity of $J_3$ may be harder because this functional is not convex.

6.2. Discretisation and Minimisation. Let us consider the penalised discrete problems

$$
z^* \in \arg \min_{z \in Z_{k,i}} \Phi_{k,i}(z), \quad \Phi_{k,i} = \Pi_k(z) + J_k(z),
$$

where $k \in \{1, 2, 3\}$, $Z_{k,i}$ is an $L$-dimensional subspace of $Z_k (L \in \{1, 2, \ldots\})$ and $\Pi_k$ is a penalty function related to
the set $M_{\ell} = P_{\ell} M$ with $P_{\ell}$ being the orthogonal projection onto $\mathcal{F}_{k,L}$. The general assumptions for $\Pi_{\ell}$ are

$$\Pi_{\ell}-\text{accretive, convex, Fréchet differentiable,} \tag{111}$$

$$\Pi_{\ell}^{1}-\text{uniformly Lipschitz continuous in } \mathcal{F}_{k,L}.$$ 

**Theorem 10.** The problem (110) has a solution.

**Proof.** The proof repeats the proof of the statement (ii) of Theorem 9, because in view of the accretivity of $\Phi_{k,L}$ a minimizing sequence is bounded and in a finite-dimensional space any bounded sequence is compact.

The Fréchet derivative of $\Phi_{k,L}$, i.e., $\Phi_{k,L}'(z) = \Pi_{\ell}(z) + J_{\ell}(z)$, is identically $\Pi_{\ell}$. Thus, by virtue of (57), (65), (82), and (90), it holds

$$\Pi_{\ell}(z) d\xi = \left< \Phi_{k,L}'(z), d\xi \right>_{\mathcal{F}_{k,L}} \forall \xi \in \mathcal{F}_{k,L},$$

where $\langle \cdot, \cdot \rangle_{\mathcal{F}_{k,L}}$ is the inner product of $\mathcal{F}_{k,L}$. In particular, the addend $J_{\ell}(z)$ is identical to the element $P_{\ell} u_{\ell}(z)$ where $u_{\ell}(z)$ is the kernel of the functional $J_{\ell}(z)$. Thus, by virtue of (57), (65), (82), and (90), it holds

$$w_{1}(z) = \sum_{i=1}^{N} \int_{0}^{T} y_{i}(t) \psi_{i}(\cdot, T_{i} - t; z) dt F_{i,1} \cdots F_{i,N} \xi_{i} \delta_{i}$$

$$\Pi_{\ell}(z) d\xi = \left< \Phi_{k,L}'(z), d\xi \right>_{\mathcal{F}_{k,L}} \forall \xi \in \mathcal{F}_{k,L},$$

where $\langle \cdot, \cdot \rangle_{\mathcal{F}_{k,L}}$ is the inner product of $\mathcal{F}_{k,L}$. In particular, the addend $J_{\ell}(z)$ is identical to the element $P_{\ell} u_{\ell}(z)$ where $u_{\ell}(z)$ is the kernel of the functional $J_{\ell}(z)$. Thus, by virtue of (57), (65), (82), and (90), it holds

$$w_{1}(z) = \sum_{i=1}^{N} \int_{0}^{T} y_{i}(t) \psi_{i}(\cdot, T_{i} - t; z) dt F_{i,1} \cdots F_{i,N} \xi_{i} \delta_{i}$$

Moreover, it holds $M_{\ell} = \{ z \in \mathcal{F}_{1,L} : \| z \| \leq \rho \}$. Define a convex penalty function $\Pi_{\ell}(z) \in C^{\alpha}[0, \infty)$ such that $\Pi_{\ell}(z) = 0$ for $\| z \| \leq \rho$ and $\Pi_{\ell}(z) = d \| z \|^2 - \rho^2$ for $\| z \| \geq \rho + \epsilon$ with some $d, \epsilon > 0$. Then $\Pi_{\ell}$ satisfies (111).

Let $k \in \{1, 2, 3\}$. Choose some initial guess $z_{0} \in \mathcal{F}_{k,L}$. Compute the approximate solutions by the gradient method

$$z_{i+1} = z_{i} - c_{i} \Phi_{k,L}'(z_{i}),$$

where $s = 0, 1, 2, \ldots$ and $c_{i} > 0$.

**Theorem 12.** Let $k \in \{1, 2\}$ and $c_{i}$ be chosen by the rule

$$\inf_{c > 0} \Phi_{k,L} \left( \xi_{i} - c \Phi_{k,L}'(z_{i}) \right) \leq \Phi_{k,L} \left( \xi_{i} - c \Phi_{k,L}'(z_{i}) \right) + \delta_{i},$$

where $\delta_{i} \geq 0$. Then $\Pi_{\ell}$ is uniformly Lipschitz continuous, $\Phi_{k,L}$ is convex, and the set $M(z_{0}) = \{ z \in \mathcal{F}_{k,L} : \Phi_{k,L}'(z) \leq \Phi_{k,L}(z_{0}) + \delta \}$ is bounded. The convexity of $\Phi_{k,L}$ follows from the convexity of its addends $\Pi_{\ell}$ and $J_{\ell}$. The boundedness of $M(z_{0})$ is a direct consequence of the accretivity of $\Phi_{k,L}$ following from the accretivity of the addend $\Pi_{\ell}$.

It remains to show the uniform Lipschitz continuity of $J_{\ell}(z)$ in $\mathcal{F}_{k,L}$ (such a property for $P_{\ell} u_{\ell}$ is assumed in (111)). Let $k = 1$. Then by (113) and $J_{\ell}(z) = P_{\ell} u_{\ell}(z)$ for any $z, \tilde{z} \in \mathcal{F}_{k,L}$, we have

$$\| J_{\ell}'(z) - J_{\ell}'(\tilde{z}) \| \leq 2\| P_{\ell} \| \| u_{\ell}(z) - u_{\ell}(\tilde{z}) \|$$

where $C_{12}$ is a constant independent of $z$ and $\tilde{z}$. Further, observing (58) and (40), the estimate (20) of Theorem 2 and $z = \omega$, we deduce

$$\| J_{\ell}'(z) - J_{\ell}'(\tilde{z}) \| \leq 2C_{12} \| u_{\ell}(\cdot, T_{i}, z) - u_{\ell}(\cdot, T_{i}, \tilde{z}) \|_{L^{2}(\Omega)}$$

where $C_{12}$ is a constant independent of $z$ and $\tilde{z}$. This proves the uniform Lipschitz continuity of $J_{\ell}$. Such a property of $J_{\ell}$ can be proved in a similar manner.

The convergence of $z_{i}$, in case $k = 3$ is an open issue. This case is more complex because IP3 is nonlinear and the Fréchet derivative of $J_{3}$, i.e., $P_{\ell} u_{\ell}$, is not uniformly Lipschitz continuous.

The quasisolutions of IP1–IP3 are not expected to be stable with respect to the noise of the data, that is, the problems...
under consideration may be ill posed. Nevertheless, from the intuitive viewpoint, a discretisation should regularise an ill-posed problem. Such a property of the discretisation has been proved in many cases [27, 28]. Alternatively, the index \( s \) of the gradient method could be used as a regularization parameter (see [29, 30]). Moreover, the addend \( \Pi_f \) can be defined to be the stabilizing term of the Tikhonov’s method instead of the penalty function, that is, \( \Pi_f = \alpha \| \z \| \), where \( \alpha > 0 \) is the regularization parameter. Such a \( \Pi_f \) satisfies (III).

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References


WEAK INVERSE PROBLEMS FOR PARABOLIC
INTEGRO-DIFFERENTIAL EQUATIONS CONTAINING
TWO KERNELS

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Abstract. An inverse problem to determine a coefficient and two kernels in a parabolic integro-differential equation is considered. A corresponding direct problem is supposed to be in the weak form. Existence of the quasi-solution is proved and issues related to Fréchet differentiation of the cost functional are treated.

1. Introduction

Inverse problems to determine coefficients and kernels in integro-differential heat equations are well-studied in the smooth case when the medium is continuous and corresponding direct problems hold in the classical sense (selection of references: [2, 4, 5, 9, 10, 12, 13, 15, 16, 17, 19]). For instance, in [10] problems to determine space-dependent coefficients by means of final over-determination of the solution of the direct problem are dealt with. This paper exploits and generalizes methods developed earlier in the usual parabolic case [3, 7].

Results are known for particular non-smooth cases, as well. For instance, identification problems for parabolic transmission problems are considered in [11] under additional smoothness assumptions in neighbourhoods of observation areas. Several papers deal with degenerate cases (see [8] and references therein). In [14] problems to reconstruct free terms and coefficients in a weak parabolic problem containing a single kernel (heat flux relaxation kernel) are considered. In particular, a new method that enables to deduce formulas for Fréchet derivatives for cost functionals of inverse problems is proposed.

In the present article we consider the inverse problem of determining two kernels and a coefficient in a parabolic integro-differential equation. The corresponding direct problem is posed in the weak form. We prove the Fréchet differentiability of the cost functional related to the inverse problem and deduce a suitable form for the Fréchet derivative in terms of an adjoint problem. In this connection we use an integrated convolutional form of the weak direct problem that enables to use test functions without classical time derivatives. Finally, we prove the existence of the quasi-solution of the inverse problem under certain restrictions.

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Inverse problems for smooth models with two kernels were formerly considered in [5, 12, 19].

2. Formal statement of problems

Let \( \Omega \) be a \( n \)-dimensional domain, where \( n \geq 1 \) and \( \Gamma = \partial \Omega \). Further, let \( \Gamma = \Gamma_1 \cup \Gamma_2 \) with \( \text{meas} \Gamma_1 \cap \Gamma_2 = 0 \), \( \text{meas} \Gamma_2 > 0 \) and either \( \Gamma_1 = \emptyset \) or \( \text{meas} \Gamma_1 > 0 \). In case \( n \geq 2 \) we assume \( \Gamma \) to be sufficiently smooth. Define

\[
\Omega_t = \Omega \times (0, t), \quad \Gamma_{1,t} = \Gamma_1 \times (0, t), \quad \Gamma_{2,t} = \Gamma_2 \times (0, t)
\]

for \( t \geq 0 \).

Let \( T > 0 \). We pose the formal direct problem: find \( u(x, t) : \Omega_T \to \mathbb{R} \) such that

\[
\begin{align*}
\nu u_t + (\mu \ast u)_t &= Au - m \ast Au + f + \nabla \cdot \phi + \varphi_t \quad \text{in } \Omega_T, \quad (2.1) \\
u u &= u_0 \quad \text{in } \Omega \times \{0\}, \quad (2.2) \\
u u &= g \quad \text{in } \Gamma_{1,T}, \quad (2.3) \\
 - \nu_A \cdot \nabla u + m \ast \nu_A \cdot \nabla u &= \partial u + h + \nu \cdot \phi \quad \text{in } \Gamma_{2,T}, \quad (2.4)
\end{align*}
\]

where

\[
Av = \sum_{i,j=1}^{n} (a_{ij} \nu_j) x_i, \quad \nu_A = \left( \sum_{j=1}^{n} a_{ij} \nu_j \right)_{|i=1,\ldots,n},
\]

\( \nu = (\nu_1, \ldots, \nu_n) \) is the outer normal of \( \Gamma_2 \), \( a_{ij}, a, u_0 : \Omega \to \mathbb{R} \), \( f, \phi : \Omega_T \to \mathbb{R} \), \( \phi : \Omega_T \to \mathbb{R}^n \), \( g : \Omega_T \to \mathbb{R} \), \( \theta : \Gamma_2 \to \mathbb{R} \), \( h : \Gamma_{2,T} \to \mathbb{R} \), \( \mu, m : (0, T) \to \mathbb{R} \) are given functions and

\[
z \ast w(t) = \int_0^t z(t - \tau) w(\tau) d\tau
\]

is the convolution with respect to the variable \( t \). In the case \( \Gamma_1 = \emptyset \), the boundary condition (2.3) is omitted. The second and third addend of the free term of the equation (2.1); i.e., \( \nabla \cdot \phi \) and \( \varphi_t \) may be singular distributions.

The problem (2.1)-(2.4) governs the heat conduction in the body \( \Omega \) filled with material with memory, where \( \mu \) and \( m \) are the relaxation kernels of the internal energy and the heat flux, respectively and \( u \) is the temperature [1, 4, 5, 18]. Then the condition (2.4) corresponds to the third kind boundary condition, namely it contains the heat flux to the co-normal direction \( -\nu_A \cdot \nabla u + m \ast \nu_A \cdot \nabla u \).

Let us formulate the inverse problem:

**IP.** Find \( a, m \) and \( \mu \) such that the solution of (2.1)-(2.4) satisfies the following final and integral additional conditions:

\[
u u = u_T \quad \text{in } \Omega \times \{T\}, \quad (2.5)
\]

\[
\int_{\Gamma_{2,j}} \kappa_j(x, \cdot u(x, \cdot)) d\Gamma = v_j \quad \text{in } (0, T), \quad j = 1, 2, \quad (2.6)
\]

where \( u_T : \Omega \to \mathbb{R} \), \( \kappa_j : \Gamma_{2,T} \to \mathbb{R} \) and \( v_j : (0, T) \to \mathbb{R} \) are prescribed functions.

**Remark 2.1.** In the case \( n = 1 \) and \( \Omega = (c, d) \), the integral \( \int_{\Gamma_{2,j}} z(x) d\Gamma \) is merely the sum \( \sum_{i=1}^{L} z(x_i) \), where \( x_i \in \Gamma_2 \subseteq (c, d) \) and \( L \) is the number of points in \( \Gamma_2 \) (i.e \( L \in (1, 2) \)). Then the conditions (2.6) read

\[
\sum_{i=1}^{L} \kappa_j(x_i, \cdot) u(x_i, \cdot) = v_j \quad \text{in } (0, T), \quad j = 1, 2. \quad (2.7)
\]
3. Results Concerning Direct Problem

Let us start by a rigorous mathematical formulation of the direct problem. Define the following functional spaces:

\[ U(\Omega_t) = C([0, t]; L^2(\Omega)) \cap L^2((0, t); W^1_2(\Omega)), \]

\[ U_0(\Omega_t) = \left\{ \eta \in U(\Omega_t) : \eta|_{\Gamma_1, t} = 0 \text{ in case } \Gamma_1 \neq \emptyset \right\}, \]

\[ T(\Omega_t) = \left\{ \eta \in L^2((0, t); W^1_2(\Omega)) : \eta \in L^2((0, t); L^2(\Omega)) \right\}, \]

\[ T_0(\Omega_t) = \left\{ \eta \in T(\Omega_t) : \eta|_{\Gamma_1, t} = 0 \text{ in case } \Gamma_1 \neq \emptyset \right\} \]

and introduce the following basic assumptions on the data of the direct problem:

\[ a_{ij} \in L^\infty(\Omega), \quad a_{ij} = a_{ji}, \quad \vartheta \in C(\bar{\Omega}), \quad \vartheta \geq 0, \quad (3.1) \]

\[ \sum_{i,j=1}^n a_{ij}(x)\lambda_i \lambda_j \geq \epsilon|\lambda|^2, \quad x \in \Omega, \quad \lambda \in \mathbb{R}^n \text{ with some } \epsilon > 0, \quad (3.2) \]

\[ a \in L^q(\Omega), \quad \text{where } q_1 = 1 \text{ if } n = 1, \quad q_1 > \frac{n}{2} \text{ if } n \geq 2, \quad (3.3) \]

\[ \mu \in L^2(0, T), \quad m \in L^1(0, T), \quad (3.4) \]

\[ u_0 \in L^2(\Omega), \quad g \in T(\Omega_T), \quad h \in L^2(\Gamma_{2,T}), \quad (3.5) \]

\[ f \in L^2((0, T); L^p(\Omega)), \quad \text{where } q_2 = 1 \text{ if } n = 1, \quad (3.6) \]

\[ q_2 \in (1, q_1) \text{ if } n = 2, \quad q_2 = \frac{2n}{n+2} \text{ if } n \geq 3, \quad (3.6) \]

\[ \phi = (\phi_1, \ldots, \phi_n) \in (L^2(\Omega_T))^n, \quad (3.7) \]

\[ \varphi \in U(\Omega_T), \text{ in case } \Gamma_1 \neq \emptyset, \quad (3.8) \]

\[ \varphi \in \partial \Omega_{\varphi}, \text{ in } \Gamma_{1,T}. \quad (3.9) \]

If we assume additional conditions \( a_{ij} \in W^{1}_{2}(\Omega), \) \( \frac{\partial}{\partial n} \phi_i \in L^2(\Omega_T), \) \( \phi_i \in L^2(\Omega_T) \) and suppose that (2.1)-(2.4) has a classical solution \( u \in L^2(\Omega_T) \) such that \( u_{t_i}, u_{x_i}, u_{x_i x_j} \in L^2(\Omega_T), \) \( i, j = 1, \ldots, n, \) then multiplying (2.1) with a test function \( \eta \in T_0(\Omega_T) \) and integrating by parts we come to the relation

\[
0 = \int_{\Omega} [(u + \mu * u)(x, T) \eta(x, T) - u_0(x) \eta(x, 0)] dx - \int_{\Omega_T} (u + \mu * u) \eta_t dx dt \\
+ \int_{\Omega_T} \left[ \sum_{i,j=1}^n a_{ij}(u_{x_j} - m * u_{x_j}) \eta_{x_i} - a(u - m * u) \eta \right] dx dt \\
+ \int_{\Omega_T} (\vartheta u + h) \eta d\Gamma dt - \int_{\Omega_T} (f \eta - \phi \cdot \nabla \eta) dx dt \\
- \int_{\Omega} [\varphi(x, T) \eta(x, T) - \varphi(x, 0) \eta(x, 0)] dx + \int_{\Omega_T} \varphi \eta_t dx dt. \quad (3.10)
\]

This relation makes sense also in a more general case when \( a_{ij}, \phi, \varphi \) satisfy (3.1), (3.7), (3.8) and \( u \in U(\Omega_T). \)

We call a weak solution of the problem (2.1)-(2.4) a function belonging to \( U(\Omega_T) \) that satisfies the relation (3.10) for any \( \eta \in T_0(\Omega_T) \) and, in case \( \Gamma_1 \neq \emptyset, \) that fulfills the boundary condition (2.3).
Theorem 3.1. Problem (2.1)–(2.4) has a unique weak solution. This solution satisfies the estimate
\[
\|u\|_{L^2(\Omega_T)} \leq C_0 \left[ \|u_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);L^2(\Omega))} + \|\phi\|_{L^2(\Gamma_{1,T})}^n \right]
\]  
\[ + \|\psi\|_{L^2(\Gamma_{1,T})} + \theta \{(g, g_t)_{L^2(\Gamma_{1,T})} + \|g_\mu\|_{L^2(\Gamma_{1,T})}\} + \|h\|_{L^2(\Gamma_{2,T})}, \]

where \( \theta = 0 \) in case \( \Gamma_1 = \emptyset \) and \( C_0 \) is a constant independent of \( u_0, f, \phi, \varphi, g, h \).

Proof. Since \( \mu \in L^2(0,T) \), the Volterra equation of the second kind
\[
\tilde{\mu} + \mu * \tilde{\mu} = -\mu \quad \text{in} \ (0,T),
\]

has a unique solution \( \tilde{\mu} \in L^2(0,T) \) [6]. We call \( \tilde{\mu} \) the resolvent kernel of \( \mu \). Further, let us consider the following problem:
\[
\tilde{u}_t = A\tilde{u} - \tilde{m} \ast \tilde{A}\tilde{u} + \tilde{f} + \varphi \cdot \tilde{\phi} \quad \text{in} \ \Omega_T,
\]
\[
\tilde{u} = \tilde{u}_0 \quad \text{in} \ \Omega \times \{0\},
\]
\[
\tilde{u} = \tilde{g} \quad \text{in} \ \Gamma_{1,T},
\]
\[
-\nu \nabla \tilde{u} + \tilde{m} \ast \nabla \tilde{u} = \varphi \tilde{u} + \tilde{\eta} \tilde{u} + \tilde{\mu} \ast \tilde{u} + \tilde{h} + \nu \cdot \tilde{\phi} \quad \text{in} \ \Gamma_{2,T},
\]

where
\[
\tilde{m} = m - \tilde{\mu} + m \ast \tilde{\mu}, \quad \tilde{f} = f + a \varphi - \tilde{m} \ast a \varphi,
\]
\[
\tilde{\phi}_t = \phi_t + \sum_{j=1}^n a_{ij} \varphi_{x_j} - \tilde{m} \ast \sum_{j=1}^n a_{ij} \varphi_{x_j},
\]
\[
\tilde{h} = h + \varphi \varphi + \varphi \tilde{\mu} \ast \varphi - \tilde{g} = g + \mu \ast g - g_\mu, \quad \tilde{u}_0 = u_0 - \varphi(\cdot, 0).
\]

By the properties of \( m \) and \( \tilde{\mu} \) we have \( \tilde{m} \in L^1(0,T) \). Further, [14, Lemma 1] yields
\[
\mathcal{U}(\Omega_T) \hookrightarrow L^2((0,T);L^n(\Omega)), \quad \text{where} \ q_1 = \infty \text{ if } n = 1,
\]
\[
q_3 > \frac{q_1 q_2}{q_1 - q_2} \text{ if } n = 2, \ q_3 = \frac{2n}{n - 2} \text{ if } n \geq 3,
\]

and
\[
\|au\|_{L^2((0,T);L^{q_1}(\Omega))} \leq C \|a\|_{L^1(\Omega)} \|u\|_{L^2((0,T);L^{q_1}(\Omega))},
\]

where \( C \) is a constant. Using the relations (3.17), (3.18), the properties of \( \tilde{m}, \tilde{\mu} \), the assumptions (3.1)–(3.8), trace theorems and the Young theorem for convolutions we obtain
\[
\tilde{d} : = (\tilde{f}, \tilde{\phi}, \tilde{u}_0, \tilde{g}, \tilde{h}) \in X
\]
\[
:= L^2((0,T);L^{q_3}(\Omega)) \times (L^2(\Omega_T))^{n} \times L^2(\Omega) \times T(\Omega_T) \times L^2(\Gamma_{2,T}),
\]

\[
\|\tilde{d}\|_X \leq \tilde{C} \|d\|_X
\]

(3.19)
where \( d = (f, \phi, u_0, g, h, \varphi, g_\varphi) \) and
\[
\tilde{K} = L^2((0,T);L^{q_3}(\Omega)) \times (L^2(\Omega_T))^{n} \times L^2(\Omega) \times T(\Omega_T) \times L^2(\Gamma_{2,T}) \times \mathcal{U}(\Omega_T) \times T(\Omega_T)
\]

and \( \tilde{C} \) is a constant. It was proved in [14, Theorem 1] that problem (2.1)–(2.4) in case \( \mu = 0 \) and \( \varphi = 0 \) has for any \( (f, \phi, u_0, g, h) \in X \) a unique weak solution and the corresponding solution operator \( \mathcal{B} \) belongs to \( L(X; \mathcal{U}(\Omega_T)) \). (Here \( X, Y \) stands for the space of linear bounded operators from a Banach space \( X \) to a Banach space.
This implies that problem (3.13)-(3.16) is equivalent in $\mathcal{U}(\Omega_T)$ to the following operator equation:

$$
\hat{u} = Q\hat{u} \quad \text{with} \quad Q\hat{u} = B(0, 0, 0, 0, \phi \hat{u} \ast \hat{u}) + B\hat{u}.
$$

(3.20)

To study this equation, we will use the inequality

$$
\|\hat{u} \ast y\|_{L^\infty(\Omega)} \leq \int_0^t |\hat{u}(t - \tau)| \|y\|_{L^\infty(\Omega)} d\tau, \quad t \in [0, T]
$$

(3.21)

that holds for any $y \in L^2(\Omega_T)$. This was proved in [14, inequality (3.12)].

Let $\hat{u}^1, \hat{u}^2 \in \mathcal{U}(\Omega_T)$, denote $v = \hat{u}^1 - \hat{u}^2$ and estimate $Q\hat{u}^1 - Q\hat{u}^2 = B(0, 0, 0, 0, \phi \hat{u} \ast v)$. To this end, fix $t \in [0, T]$ and define

$$
P_{\hat{u}, t}w = \begin{cases}
w & \text{in } \Gamma_{2,t}^c, \\
0 & \text{in } \Gamma_{2,t}^c \setminus \Gamma_{2,t}
\end{cases}
$$

(3.22)

for $w : \Gamma_{2,T} \rightarrow \mathbb{R}$. Due to the causality, we have $B(0, 0, 0, 0, P_{\hat{u}} \phi \hat{u} \ast v)(x, \tau) = B(0, 0, 0, \phi \hat{u} \ast v)(x, \tau)$ for any $(x, \tau) \in \Omega_t$. Since $B \in L(L^1(\mathcal{U}(\Omega_T)))$, the continuity of $\hat{u}$, the trace theorem and the inequality (3.21) with $y = v, v_{x_i}, t = 1, \ldots, n$, it follows that

$$
\|Q\hat{u}^1 - Q\hat{u}^2\|_{\mathcal{U}(\Omega_T)} = \|B(0, 0, 0, 0, \phi \hat{u} \ast v)\|_{\mathcal{U}(\Omega_T)}
= \|B(0, 0, 0, 0, P_{\hat{u}} \phi \hat{u} \ast v)\|_{\mathcal{U}(\Omega_T)}
\leq \|B(0, 0, 0, 0, 0)\|_{\mathcal{U}(\Omega_T)}
\leq \|B\|_{L^1(\mathcal{U}(\Omega_T))} \|\phi \hat{u} \ast v\|_{L^1(\Omega_t)}
\leq C_1 \|\phi \hat{u} \ast v\|_{L^1(\Omega_t)}
\leq C_2 \int_0^t |\hat{u}(t - \tau)| \|v\|_{L^2(\Omega_t)} d\tau
$$

(3.23)

with some constants $C_1$ and $C_2$. Let us define the weighted norm in $\mathcal{U}(\Omega_T)$:

$$
\|v\|_{\sigma} = \sup_{0 < \tau < T} e^{-\tau}\|v\|_{\mathcal{U}(\Omega_t)}, \quad \sigma \geq 0.
$$

In view of (3.23) and $\|\hat{u}\|_{\mathcal{U}(\Omega_T)} \in L^2((0, T); W_2^1(\Omega))$ we get

$$
\|Q\hat{u}^1 - Q\hat{u}^2\|_{\sigma} \leq C_3 \sup_{0 < \tau < T} e^{-\tau} \int_0^t |\hat{u}(t - \tau)| \|v\|_{\mathcal{U}(\Omega_t)} d\tau
= C_3 \sup_{0 < \tau < T} \int_0^t e^{-\tau(t - \tau)} \|\hat{u}(t - \tau)| \|v\|_{\mathcal{U}(\Omega_t)} d\tau
\leq C_4 \int_0^T e^{-\sigma \tau} |\hat{u}(s)| ds \sup_{0 < \tau < T} e^{-\sigma \tau} \|v\|_{\mathcal{U}(\Omega_T)}
= C_5 \int_0^T e^{-\sigma \tau} |\hat{u}(s)| ds \|v\|_{\sigma}
$$

(3.24)

with some constant $C_3$. By the dominated convergence theorem, $\int_0^T e^{-\sigma \tau} |\hat{u}(s)| ds \rightarrow 0$ as $\sigma \rightarrow \infty$. Thus, there exists $\sigma_0$ such that

$$
C_3 \int_0^T e^{-\sigma_0 \tau} |\hat{u}(s)| ds \leq \frac{1}{2}.
$$

Therefore, $\|Q\hat{u}^1 - Q\hat{u}^2\|_{\sigma_0} \leq \frac{1}{2} \|Q\hat{u}^1 - Q\hat{u}^2\|_{\sigma_0}$. The operator $Q$ is a contraction in $\mathcal{U}(\Omega_T)$. This implies that (3.13)-(3.16) has a unique weak solution in $\mathcal{U}(\Omega_T)$. 

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Moreover, observing (3.20) and the relation \( Q = B \tilde{u} \), for the solution of (3.13)–(3.16) we obtain the estimate
\[
\| \tilde{u} \|_{\sigma_0} = \| Q \tilde{u} - Q \tilde{u} + Q \tilde{u} \|_{\sigma_0} \leq \| Q \tilde{u} - Q \tilde{u} \|_{\sigma_0} + \| B \tilde{d} \|_{\sigma_0} \leq \frac{1}{2} \| \tilde{u} \|_{\sigma_0} + \| B \tilde{d} \|_{\sigma_0}
\]
which implies
\[
\| \tilde{u} \|_{\sigma_0} \leq 2 \| B \tilde{d} \|_{\sigma_0} \leq 2 \| B \tilde{d} \|_{\mathcal{U}(\Omega_T)} \leq 2 \| B \| \| \tilde{d} \|_{X}.
\]
Observing the relation \( e^{-\sigma_0 T} \| \tilde{u} \|_{\mathcal{U}(\Omega_T)} \leq \| \tilde{u} \|_{\sigma_0} \) and (3.19) we arrive at the estimate
\[
\| \tilde{u} \|_{\mathcal{U}(\Omega_T)} \leq C_4 \| \tilde{d} \|_{X} \tag{3.24}
\]
with a constant \( C_4 \).

Further, let us define
\[
u = \tilde{u} + \varphi = \mu \ast (\tilde{u} + \varphi).
\]
Then \( \tilde{u} \) is expressed in terms of \( u \) as
\[
\tilde{u} = u + \mu \ast (\tilde{u} + \varphi).
\]
One can immediately check that the implications \( u \in \mathcal{U}(\Omega_T) \Leftrightarrow \tilde{u} \in \mathcal{U}(\Omega_T) \) are valid. Moreover, it is easy to see that \( \tilde{u} \) is a weak solution of (3.13)–(3.16) if and only if \( u \) is a weak solution of (2.1)–(2.4). In view of the above-presented arguments we can conclude that (2.1)–(2.4) has a unique weak solution. From (3.25) we obtain
\[
\| u \|_{\mathcal{U}(\Omega_T)} \leq C_5 (\| \tilde{u} \|_{\mathcal{U}(\Omega_T)} + \| \varphi \|_{\mathcal{U}(\Omega_T)})
\]
with a constant \( C_5 \). This with (3.24) implies (3.11). The proof is complete. \( \square \)

It is possible to give an equivalent form to the relation (3.10) that does not contain the derivative of the test function with respect to \( t \). Namely, the following theorem holds.

**Theorem 3.2.** The function \( u \in \mathcal{U}(\Omega_T) \) satisfies the relation (3.10) for any \( \eta \in \mathcal{T}_0(\Omega_T) \) if and only if it satisfies the relation
\[
0 = \int_{Q} (u + \mu \ast u - \varphi) \ast \eta dx + \int_{Q} \int_{0}^{t} (u_0(x) - \varphi(x, 0)) \eta(x, \tau) d\tau dx
\]
\[
+ \int_{Q} \frac{1}{\gamma} \left[ \sum_{i,j=1}^{n} a_{ij}(u_{x_i}, -m \ast u_{x_j}) \ast \eta_{x_i} - a(u - m \ast u) \ast \eta \right] dx
\]
\[
+ \int_{\Gamma_1} 1 \ast (\theta u + h) \ast \eta d\Gamma - \int_{Q} \frac{1}{\gamma} \left( f - \sum_{i=1}^{n} \phi_i \ast \eta_{x_i} \right) dx,
\]
for any \( t \in [0, T] \) and \( \eta \in \mathcal{T}_0(\Omega_T) \).

**Proof.** It is analogous to the proof of [14, Theorem 2] that considers the case \( \varphi = 0, \mu = 0 \). We have only to replace \( u \) by \( \tilde{u} = u + \mu \ast u - \varphi \) in the term \( K_1(t) \) appearing in formulas [14, (3.19), (3.20)] to get the desired result. \( \square \)

**Remark 3.3.** Theorems 3.1 and 3.2 remain valid also in the case \( \Gamma_2 = \emptyset \). In this case the terms \( \| h \|_{L_{2,T}} \) and \( \int_{P_t} 1 \ast (\theta u + h) \ast \eta d\Gamma \) are missing in (3.11) and (3.27), respectively.
4. Quasi-solution of IP. Fréchet derivative of cost functional

Assume that $n \in \{1, 2, 3\}$. Moreover, let us set $q_1 = 2$ if $n = 2$. Then any coefficient $a$ that belongs to $L^2(\Omega)$ satisfies (3.3). For the weight functions $\kappa_j$ we assume that

$$\kappa_j \in L^\infty((0, T); L^2(\Gamma_2)), \quad j = 1, 2.$$  (4.1)

In the case $n = 1$ this assumption is simply $\kappa(x, \cdot) \in L^\infty(0, T)$, $x \in \Gamma_2 \subseteq \{e; d\}$. According to Theorem 3.1, $u \in \mathcal{U}(\Omega_T)$, thus $u(\cdot, T) \in L^2(\Omega)$, and the condition (2.5) is well-defined for $u_T \in L^2(\Omega)$. Moreover, by a trace theorem we have $u \in L^2(\Gamma_3, T)$. This implies that $\int_{\Gamma_3} \kappa_j(x, \cdot)u(x, \cdot)d\Gamma \in L^2(0, T)$, $j = 1, 2$, hence the condition (2.6) is well-defined for $v_j \in L^2(0, T)$, $j = 1, 2$.

Let $M \subseteq Z := L^2(\Omega) \times (L^2(0, T))^2$. We call the quasi-solution of IP in the set $M$ an element $z^* \in \text{arg}\min_{z \in M} J(z)$, where $J$ is the cost functional

$$J(z) = \|u(\cdot, T; z) - u_T\|^2_{L^2(\Omega)} + \sum_{j=1}^2 \int_{\Gamma_3} \kappa_j(x, \cdot)u(x, \cdot)u(x, \cdot)\Gamma - v_j(\cdot)\|_{L^2(0, T)}^2$$

and $u(x, t; z)$ is the weak solution of the direct problem (2.1)-(2.4) corresponding to given $z = (a, m, \mu)$. In case $n = 1$ the integral $\int_{\Gamma_3} \kappa_j(x, t)u(x, t; z)d\Gamma$ in the definition of $J$ is replaced by $\sum_{j=1}^2 \kappa_j(x, t)u(x, t; z)$.

**Theorem 4.1.** The functional $J$ is Fréchet differentiable in $Z$ and $J'(z)\Delta z$

$$= 2 \int_{\Omega} [u(x, T; z) - u_T(z)] \Delta u(x, T)dx$$

$$+ 2 \sum_{j=1}^2 \int_0^T \left[ \int_{\Gamma_3} \kappa_j(y, t)u(y, t; z)d\Gamma - v_j(t) \right] \int_{\Gamma_3} \kappa_j(x, t)\Delta u(x, t)d\Gamma dt,$$

where $\Delta z = (\Delta a, \Delta m, \Delta \mu) \in Z$ and $\Delta u \in \mathcal{U}(\Omega_T)$ is the $z$- and $\Delta z$-dependent weak solution of the following problem:

$$\Delta u_t + (\mu \ast \Delta u)_t = \Delta u - m \ast \Delta u + \Delta a[u - m \ast u] - \Delta m \ast \Delta u$$

$$- \nabla \cdot \left( \Delta m \ast \sum_{j=1}^n a_{ij}u_s \right) - (\Delta m \ast \Delta u)_t \quad \text{in } \Omega_T,$$

$$\Delta u = 0 \quad \text{in } \Omega \times \{0\},$$

$$\Delta u = 0 \quad \text{in } \Gamma_1, T,$$

$$- \nu_A \cdot \nabla u + m \ast \nu_A \cdot \nabla u$$

$$= \partial \Delta u - \nu \left[ \Delta m \ast \sum_{j=1}^n a_{ij}u_s \right] \quad \text{in } \Gamma_2, T.$$  (4.3)

**Proof.** Denote $\Delta u = u(x, t; z + \Delta z) - u(x, t; z)$ and define $\Delta u = \Delta u - \Delta u$. Then we can represent the difference of $J$ as follows:

$$J(z + \Delta z) - J(z) = \text{RHS} + \Theta,$$

where RHS is the right-hand side of the equality (4.2) and

$$\Theta = 2 \int_{\Omega} [u(x, T) - u_T(x)] \Delta u(x, T)dx$$

(4.7)
\[ + 2 \sum_{j=1}^{2} \int_0^T \left[ \int_{\Gamma_2} \kappa_j(y, t) u(y, t) \, d\Gamma - v_j(t) \right] \int_{\Gamma_3} \kappa_j(x, t) \Delta u(x, t) \, d\Gamma \, dt \\
+ \int_{\Gamma_3} \left\{ (\Delta u + \Delta \tilde{u})(x, T) \right\}^2 \, dx \\
+ 2 \sum_{j=1}^{2} \int_0^T \left\{ \int_{\Gamma_2} \kappa_j(x, t) (\Delta u + \Delta \tilde{u})(x, t) \, d\Gamma \right\}^2 \, dt. \]

Let us study problem (4.3)-(4.6). To this end we estimate the terms in the right-hand side of (4.3). Observing the relations \( u \in U(\Omega_T), (3.17), (3.18), L^2(\Omega) \hookrightarrow L^\infty(\Omega) \) and using the Young and Cauchy inequalities we deduce
\[
\|\Delta a[u - m \ast u] - \Delta m \ast au\|_{L^2(0,T);L^2(\Omega))} \\
\leq c_1 \|u\|_{L^2(\Omega_T)} \left[ (1 + \|m\|_{L^2(0,T)}) \|\Delta a\|_{L^2(\Omega)} + \|a\|_{L^2(\Omega)} \|\Delta m\|_{L^2(0,T)} \right] \\
\leq c_2(z, u) \|\Delta z\|, \tag{4.8}
\]
where \( c_1 \) is a constant, \( c_2 \) is a coefficient depending on \( z = (a, m, \mu) \), and \( \| \cdot \| \) denotes the norm in \( Z \). Taking the boundeness of \( a_{ij} \) into account we similarly get
\[
\|\Delta m \ast \sum_{j=1}^{n} a_{ij} u_{x_j} \|_{L^1(\Omega_T)^n} \leq c_3 \|u\|_{U(\Omega_T)} \|\Delta m\|_{L^1(0,T)} \tag{4.9}
\]
with a constant \( c_3 \). Next let us estimate the term \( \Delta \mu \ast u \) at the right-hand side of (4.3). Since \( u \in C([0,T]; L^2(\Omega)) \) and \( \Delta \mu \in L^2(0,T) \), it is easy to check that \( \Delta \mu \ast u \in C([0,T]; L^2(\Omega)) \) and \( \|\Delta \mu \ast u\|_{C([0,T]; L^2(\Omega))} \leq T^{1/2} \|u\|_{C([0,T]; L^2(\Omega))} \|\Delta \mu\|_{L^2(0,T)} \). Similarly, \( \|\Delta \mu \ast u\|_{L^2(0,T);W^2_2(\Omega))} \leq T^{1/2} \|u\|_{L^2(0,T);W^2_2(\Omega))} \|\Delta \mu\|_{L^2(0,T)} \). Taking these estimates together, we have
\[
\|\Delta \mu \ast u\|_{U(\Omega_T)} \leq T^{1/2} \|u\|_{U(\Omega_T)} \|\Delta \mu\|_{L^2(0,T)}. \tag{4.10}
\]
Since \( u = g \) in \( \Gamma_{1,T} \), we find that
\[
\Delta \mu \ast u = \Delta \mu \ast g \quad \text{in} \quad \Gamma_{1,T}. \tag{4.11}
\]
Using the assumption \( g \in T(\Omega_T) \) and the Young and Cauchy inequalities again, we obtain
\[
\|\Delta \mu \ast g\|_{T(\Omega_T)} = \|\Delta \mu \ast g\|_{L^2(0,T);W^2_2(\Omega))} + \|\Delta \mu \ast g\|_{L^2(0,T);L^2(\Omega)} \\
= \|\Delta \mu \ast g\|_{L^2(0,T);W^2_2(\Omega))} + \|\Delta \mu \ast g\|_{L^2(0,T);L^2(\Omega)} \\
+ \|\Delta u\|_{L^2(0,T);L^2(\Omega)} \\
\leq c_4 \|\Delta \mu\|_{L^2(0,T)} \tag{4.12}
\]
with a constant \( c_4 \). Relations (4.8)-(4.12) show that Theorem 3.1 holds for problem (4.3)-(4.6), hence it has a unique weak solution \( \Delta u \in U(\Omega_T) \). Using the estimate (3.11) for the solution of this problem we obtain
\[
\|\Delta u\|_{U(\Omega_T)} \\
\leq C_0 \left[ \|\Delta a[u - m \ast u] + \Delta m \ast au\|_{L^2(0,T);L^\infty(\Omega))} + \|\Delta m \ast \sum_{j=1}^{n} a_{ij} u_{x_j} \|_{L^2(\Omega_T)^n} + \|\Delta \mu \ast u\|_{U(\Omega_T)} + \theta \|\Delta \mu \ast g\|_{T(\Omega_T)} \right] \\
\leq c_5(z, u) \|\Delta z\|. \tag{4.13}
\]
with a coefficient $c_5$ depending on $z, u$.

The function $\tilde{\Delta} u$ satisfies the problem
\[
\tilde{\Delta} u + (\mu \ast \tilde{\Delta} u)_t = \lambda \tilde{\Delta} u - m \ast \tilde{\Delta} u + f + \tilde{\nabla} \cdot \phi + \nabla \cdot \tilde{\phi} + \varphi_t + \tilde{\varphi}_t \quad \text{in } \Omega_T,
\]
\[
\tilde{\Delta} u = 0 \quad \text{in } \Omega \times \{0\},
\]
\[
\tilde{\Delta} u = 0 \quad \text{in } \Gamma_{1,T},
\]
\[
-\nu_A \cdot \nabla \tilde{\Delta} u + m \ast \nu_A \cdot \nabla \tilde{\Delta} u = \partial \tilde{\Delta} u + \nu \cdot \phi + \nu \cdot \tilde{\phi} \quad \text{in } \Gamma_{2,T},
\]
where
\[
f = \Delta a \tilde{\Delta} u - (m + \Delta m) \ast \Delta a \tilde{\Delta} u - \Delta m \ast \Delta a \tilde{\Delta} u - \Delta m \ast a \tilde{\Delta} u,
\]
\[
\tilde{f} = \Delta a \tilde{\Delta} u - (m + \Delta m) \ast \Delta a \tilde{\Delta} u - \Delta m \ast a \tilde{\Delta} u,
\]
\[
\phi = -\Delta m \ast \sum_{j=1}^{n} a_{ij} \Delta u_{x_j}, \quad \tilde{\phi} = -\Delta m \ast \sum_{j=1}^{n} a_{ij} \tilde{\Delta} u_{x_j},
\]
\[
\varphi = -\Delta \mu \ast \Delta u, \quad \tilde{\varphi} = -\Delta \mu \ast \tilde{\Delta} u.
\]

Similarly to (4.8)–(4.10) we deduce the following estimates:
\[
\|f\|_{L^2(0,T; L^2(\Omega))} \leq c_6 \left\{ (1 + \|m\|_{L^2(0,T)} + \|\Delta m\|_{L^2(0,T)} + \|\Delta a\|_{L^2(\Omega)} + \|a\|_{L^2(\Omega)}) \right\}
\]
\[
+ \|u\|_{L^2(\Omega_T)} \|\Delta m\|_{L^2(\Omega_T)} \|\Delta a\|_{L^2(\Omega)} + \|a\|_{L^2(\Omega)} \|\Delta m\|_{L^2(0,T)} \|\Delta u\|_{L^2(\Omega_T)} \}
\]
\[
\leq c_7(z, u) \left\{ \|\Delta z\| + \|\Delta z\|^2 \right\} \|u\|_{L^2(\Omega_T)} + \|\Delta z\| \},
\]
\[
\|\tilde{f}\|_{L^2(0,T; L^2(\Omega_T))} \leq c_8(z) \left\{ \|\Delta z\| + \|\Delta z\|^2 \right\} \|\tilde{\Delta} u\|_{L^2(\Omega_T)},
\]
\[
\|\phi\|_{L^2(\Omega_T)} \leq c_9 \|\Delta z\| \|\Delta u\|_{L^2(\Omega_T)},
\]
\[
\|\tilde{\phi}\|_{L^2(\Omega_T)} \leq c_9 \|\Delta z\| \|\tilde{\Delta} u\|_{L^2(\Omega_T)},
\]
\[
\|\varphi\|_{L^2(\Omega_T)} \leq T^{1/2} \|\Delta z\| \|\Delta u\|_{L^2(\Omega_T)},
\]
\[
\|\tilde{\varphi}\|_{L^2(\Omega_T)} \leq T^{1/2} \|\Delta z\| \|\tilde{\Delta} u\|_{L^2(\Omega_T)}
\]
with some coefficients $c_6, \ldots, c_9$. Moreover, since $\Delta u = \tilde{\Delta} u = 0$ in $\Gamma_1, T$, we have $\varphi = \tilde{\varphi} = 0$ in $\Gamma_1, T$. Applying the estimate (3.11) to the solution of the problem (4.14)–(4.17) we get
\[
\|\tilde{\Delta} u\|_{L^2(\Omega_T)} \leq c_{10}(z, u) \left\{ \|\Delta z\| + \|\Delta z\|^2 \right\} \left\{ \|\Delta u\|_{L^2(\Omega_T)} + \|\tilde{\Delta} u\|_{L^2(\Omega_T)} \right\} + \|\Delta z\|^2 \}
\]
with a coefficient $c_{10}$. Provided $\|\Delta z\|$ is sufficiently small; i.e., $\|\Delta z\| + \|\Delta z\|^2 \leq \frac{1}{2c_{10}(z, u)}$, we have
\[
\|\tilde{\Delta} u\|_{L^2(\Omega_T)} \leq 2c_{10}(z, u) \left\{ \|\Delta z\| + \|\Delta z\|^2 \right\} \|\Delta u\|_{L^2(\Omega_T)} + \|\Delta z\|^2 \}
\]
Due to (4.13), this yields
\[
\|\tilde{\Delta} u\|_{L^2(\Omega_T)} \leq c_{11}(z, u) \left[ \|\Delta z\|^2 + \|\Delta z\|^3 \}
\]
with a coefficient $c_{11}$.
In view of (4.13), (4.18) and the assumption \( \kappa_j \in L^\infty((0, T); L^2(\Gamma_2)) \) the right-hand side of (4.2) RHS and the quantity \( \Theta \) satisfy the estimates

\[
\text{st}|\text{RHS}| \leq c_{12}(z, u)\|\Delta z\|, \quad |\Theta| \leq c_{13}(z, u) \sum_{i=2}^6 \|\Delta z\|_i, \quad (4.19)
\]

st where \( c_{12} \) and \( c_{13} \) are some coefficients. Moreover, RHS is linear with respect to \( \Delta z \). This with (4.7) shows that \( J \) is Fréchet differentiable in \( Z \) and \( J'(\Delta z) \) equals RHS.

\[\square\]

**Theorem 4.2.** Assume \( g = 0 \). Then the Fréchet derivative of \( J \) admits the form

\[
J'(\Delta z) = \int_\Omega \gamma_1(x)\Delta a(x)dx + \int_0^T \gamma_2(t)\Delta m(t)dt + \int_0^T \gamma_3(t)\Delta \mu(t)dt, \quad (4.20)
\]

where

\[
\gamma_1(x) = [(u - m \ast u) \ast \psi](x, T), \quad (4.21)
\]

\[
\gamma_2(t) = -\int_\Omega [(u \ast \psi + \sum_{i,j=1}^n a_{ij}\psi_{x_i} \ast u_{x_j})](x, T - t)dx, \quad (4.22)
\]

\[
\gamma_3(t) = \int_\Omega \sum_{i,j=1}^n a_{ij}\psi_{x_i} \ast u_{x_j} + \sum_{i,j=1}^n a_{ij}\psi_{x_i} \ast u_{x_j} \ast [\mu - m \ast \mu]\] \]

[continued]

\[\text{where} \quad \mu \text{ is the solution of (3.12), } u(x, t) = u(x, t; z) \text{ and } \psi \in \mathcal{U}(\Omega_T) \text{ is the } z\text{-dependent weak solution of the following “adjoint” problem:}
\]

\[
\Delta \psi_t + (\mu \ast \Delta \psi)_t = \Lambda \Delta \psi - m \ast \Lambda \Delta \psi \quad \text{in } \Omega_T, \quad (4.24)
\]

\[
\Delta \psi = 2[u(\cdot, T) - u_T] \quad \text{in } \Omega \times \{0\}, \quad (4.25)
\]

\[
\Delta \psi = 0 \quad \text{in } \Gamma_1, \quad (4.26)
\]

\[
-\nu_A \cdot \nabla \Delta \psi + m \ast \nu_A \cdot \nabla \Delta \psi = \partial \Delta \psi + h^s \quad \text{in } \Gamma_2, \quad (4.27)
\]

where

\[
h^s(x, t) = -2 \sum_{j=1}^2 \kappa_j(x, T - t) \int_{\Gamma_2} \kappa_j(y, T - t)u(y, T - t)dy - v_j(T - t). \quad (4.28)
\]
Proof. Define $\Delta w = \Delta u + \Delta \mu * u + \tilde{\mu} * \Delta \mu * u$. Since $u, \Delta u \in \mathcal{U}(\Omega_T)$, we have $\Delta w \in \mathcal{U}(\Omega_T)$. Moreover, using (3.12) it is easy to see that $\Delta u + \mu * \Delta u + \Delta \mu * u = \Delta w + \mu * \Delta w$. Using this relation for the time derivatives in (4.3) and the equality $\Delta u = \Delta w - \Delta \mu * u - \tilde{\mu} * \Delta \mu * u$ for other terms containing $\Delta u$ in (4.3)-(4.6) we see that $\Delta w$ is the weak solution of the problem

$$\begin{align*}
\Delta w + (\mu * \Delta w)_t &= A \Delta w - m * A \Delta w + f^t + \nabla \cdot \phi^t \quad \text{in} \ \Omega_T, \\
\Delta w &= 0 \quad \text{in} \ \Omega \times \{0\}, \\
\Delta w &= 0 \quad \text{in} \ \Gamma_{1,T}, \\
-\nu_A \cdot \nabla \Delta u + m * \nu_A \cdot \nabla \Delta u &= \partial \Delta u + h^t + \nu \cdot \phi^t \quad \text{in} \ \Gamma_{2,T},
\end{align*}$$

where

$$f^t = \Delta a[u - m * u] - a \Delta m * u - a \Delta \mu * u - a \mu * u \cdot [\tilde{\mu} - m - m \cdot \tilde{\mu}],$$

$$\phi^t = (\phi_1^t, \ldots, \phi_n^t),$$

$$\phi_i^t = -\Delta m \cdot \sum_{j=1}^n a_{ij} u_{x_j} - \Delta \mu \cdot \sum_{j=1}^n a_{ij} u_{x_j} - \Delta \mu \cdot \sum_{j=1}^n a_{ij} u_{x_j} \cdot [\tilde{\mu} - m - m \cdot \tilde{\mu}],$$

$$h^t = -\partial \Delta \mu * [u + \tilde{\mu} * u].$$

Let us write the weak form (3.27) for the problem for $\Delta w$ and use the test function $\eta = \psi$. Then we obtain

$$0 = \int_\Omega (\Delta w + \mu * \Delta w) * \psi dx + \int_\Omega 1 * \left( \sum_{i,j=1}^n a_{ij} \Delta w_{x_j} - m * \Delta w_{x_j} \right) * \psi_{x_i},$$

$$-\alpha (\Delta w - m * \Delta w) * \psi dx + \int_{\Gamma_2} 1 * (\partial \Delta w + h^t) * \psi d\Gamma$$

where

$$n = \int_\Omega f^t * \psi - \sum_{i=1}^n \phi_i^t * \psi_{x_i} dx.$$ 

Next we write the weak form (3.27) for the problem for $\psi$ and use the test function $\eta = \Delta w$ to get

$$0 = \int_\Omega (\psi + \mu * \psi) * \Delta w dx - 2 \int_\Omega \int_0^t u(x,T) - u_T(x) | \Delta w(x,T)| dx dt$$

$$+ \int_\Omega 1 * \left[ \sum_{i,j=1}^n a_{ij} (\psi_{x_j} - m * \psi_{x_j}) \Delta w_{x_i} - \alpha (\psi - m * \psi) * \Delta w \right] dx$$

$$+ \int_{\Gamma_2} 1 * (\partial \psi + h^t) * \Delta w d\Gamma.$$

Subtracting (4.36) from (4.37), differentiating with respect to $t$ and setting $t = T$ we have

$$2 \int_\Omega [u(x,T) - u_T(x)] | \Delta w(x,T)| dx dt - \int_{\Gamma_2} (h^t * \Delta w)(x,T) d\Gamma$$

$$= \int_\Omega (f^t * \psi - \sum_{i=1}^n \phi_i^t * \psi_{x_i})(x,T) dx - \int_{\Gamma_2} (h^t * \psi)(x,T) d\Gamma.$$
Observing the relations $\Delta w = \Delta u + \Delta \mu \ast u + \bar{\mu} \ast \Delta \mu \ast u$, (4.28) and (4.2) we obtain the formula

$$J'(z) \Delta z = \int_\Omega \left( f^T \ast \psi - \sum_{i=1}^n \phi_i^T \ast \psi z_i \right) (x, T) dx - \int_{\Gamma^1} (h^T \ast \psi)(x, T) d\Gamma - 2 \int_{\Omega} [u(x, T) - u_T(x)] [\{\Delta \mu + \bar{\mu} \ast \Delta \mu \ast u\}(x, T) dx$$

$$- 2 \sum_{j=1}^2 \int_0^T \left[ \int_{\Gamma^2} \kappa_j(y, t) u(y, t; z) d\Gamma - v_j(t) \right]$$

$$\times \int_{\Gamma^2} \kappa_j(x, t; \{\Delta \mu + \bar{\mu} \ast \Delta \mu \ast u\}(x, t) d\Gamma dt.$$  

Rearranging the terms yields (4.20) with (4.21)-(4.23).

The formula (4.20) shows that the vector $(\gamma_1, \gamma_2, \gamma_3)$ is a representation of $J'(z)$ in the space $Z$. It can be used in gradient-type minimization algorithms (cf. [13, 14]).

5. EXISTENCE OF QUASI-SOLUTIONS

Theorem 5.1. Let $M$ be compact. Then $IP$ has a quasi-solution in $M$.

Proof. It coincides with the proof of [14, Theorem 7 (ii)]. We use the continuity of $J$ that is a consequence of the Fréchet differentiability of $J$ proved in the previous section.

Theorem 5.2. Let $n = 1$, $\Omega = (c, d)$, $\varphi = g_\varphi = 0$, $g(x, 0) = 0$ and $M$ be bounded, closed and convex. Then $IP$ has a quasi-solution in $M$.

Proof. This theorem follows from Weierstrass existence theorem [20] provided we are able to show that $J$ is weakly sequentially lower semi-continuous in $M$. We will prove that $J$ is in fact weakly sequentially continuous in $M$.

Let us choose some sequence $z_k = (a_k, m_k, \mu_k) \in M$ such that $z_k \rightharpoonup z = (a, m, \mu) \in M$. Then it is easy to see that $a_k \rightharpoonup a$ in $L^2(c, d)$ and $m_k \rightharpoonup m$, $\mu_k \rightharpoonup \mu$ in $L^2(0, T)$. As in the proof of Theorem 3.1, let $\hat{\mu} \in L^2(0, T)$ be the solution of (3.12). Similarly, let $\hat{\mu}_k \in L^2(0, T)$ be the solution of the equation $\hat{\mu}_k + \mu_k \ast \hat{\mu}_k = -\mu_k$ in $(0, T)$. Let us show that $\hat{\mu}_k \rightharpoonup \hat{\mu}$ in $L^2(0, T)$. To this end we firstly verify the boundedness of the sequence $\hat{\mu}_k$. Multiplying the equation of $\hat{\mu}_k$ by $e^{-t\sigma}$, $\sigma > 0$, and estimating by means of the Young and Cauchy inequalities we obtain

$$\|e^{-t\sigma} \hat{\mu}_k\|_{L^2(0, T)} \leq \|e^{-t\sigma} \mu_k \ast e^{-t\sigma} \hat{\mu}_k\|_{L^2(0, T)} + \|e^{-t\sigma} \mu_k\|_{L^2(0, T)}$$

$$\leq \|e^{-t\sigma} \mu_k\|_{L^2(0, T)} \|e^{-t\sigma} \hat{\mu}_k\|_{L^2(0, T)} + \|e^{-t\sigma} \mu_k\|_{L^2(0, T)}$$

$$\leq \|e^{-t\sigma} \hat{\mu}_k\|_{L^2(0, T)} \|\mu_k\|_{L^2(0, T)} \|e^{-t\sigma} \hat{\mu}_k\|_{L^2(0, T)} + \|e^{-t\sigma} \mu_k\|_{L^2(0, T)}.$$ 

Observing that $\|e^{-t\sigma}\|_{L^2(0, T)} \leq 1/\sqrt{2\sigma}$ and choosing $\sigma = \sigma_1 = 2\sup \|\mu_k\|_{L^2(0, T)}^2$ we get

$$\|e^{-t\sigma} \hat{\mu}_k\|_{L^2(0, T)} \leq 2\|e^{-t\sigma} \mu_k\|_{L^2(0, T)} \Rightarrow \|\hat{\mu}_k\|_{L^2(0, T)} \leq 2e^{-\sigma_1 T} \sup \|\mu_k\|_{L^2(0, T)}.$$ 

This shows that the sequence $\hat{\mu}_k$ is bounded. The difference $\hat{\mu}_k - \hat{\mu}$ can be expressed as

$$\hat{\mu}_k - \hat{\mu} = - (\mu_k - \mu) - v_k \ast (\mu_k - \mu),$$
where \( v_k = \tilde{\mu} + \tilde{\mu}_k + \tilde{\mu} * \tilde{\mu}_k \) is a bounded sequence in \( L^2(0, T) \). Denote by \( \langle \cdot, \cdot \rangle \) the inner product in \( L^2(0, T) \). With an arbitrary \( \zeta \in L^2(0, T) \) we have

\[
\langle \tilde{\mu}_k - \tilde{\mu}, \zeta \rangle = -\langle \mu_k - \mu, \zeta \rangle - N_k, \quad N_k = \int_0^T v_k(\tau) \int_0^{T-\tau} (\mu_k - \mu)(s)\zeta(\tau + s)dsd\tau. \tag{5.1}
\]

Since \( \zeta(\tau + s) \in L^2(0, T - \tau) \) for \( \tau \in (0, T) \), it holds \( \int_0^{T-\tau} (\mu_k - \mu)(s)\zeta(\tau + s)ds \to 0 \) for \( \tau \in (0, T) \). Moreover, since \( \mu_k \) is bounded in \( L^2(0, T) \), the sequence of \( \tau \)-dependent functions \( \int_0^{T-\tau} (\mu_k - \mu)(s)\zeta(\tau + s)ds \) is bounded by a constant. By the Cauchy inequality and the dominated convergence theorem, we find

\[
|N_k| \leq \|v_k\|_{L^2(0, T)} \|\int_0^{T-\tau} (\mu_k - \mu)(s)\zeta(\cdot + s)ds\|_{L^2(0, T)} \to 0.
\]

Thus, from (5.1), in view of \( \mu_k \to \mu \), we obtain \( \tilde{\mu}_k \to \tilde{\mu} \).

Let us define

\[
\tilde{u} = u + \mu * u, \quad \tilde{u}_k = u_k + \mu_k * u_k,
\]

where \( u = u(x, t; z) \) and \( u_k = u(x, t; \tilde{z}_k) \) are the weak solutions of (2.1)-(2.4) corresponding to the vectors \( z \) and \( \tilde{z}_k \), respectively. The relations \( u, u_k \in \mathcal{U}(\Omega_T) \) and \( \mu, \mu_k \in L^2(0, T) \) imply \( \tilde{u}, \tilde{u}_k \in \mathcal{U}(\Omega_T) \). Observing the definitions of the resolvent kernels \( \tilde{\mu} \) and \( \tilde{\mu}_k \) we deduce

\[
u - u = \tilde{u} - \tilde{\mu} * \tilde{u}, \quad u_k = \tilde{u}_k + \tilde{\mu}_k * \tilde{u}_k,
\]

\[
u_k - u = \tilde{u}_k - \tilde{\mu}_k * (\tilde{u}_k - \tilde{u}) + (\tilde{\mu}_k - \tilde{\mu}) * \tilde{u}.
\]

In view of the latter relation we express the difference of values of the functional \( J \) as follows:

\[
J(z_k) - J(z) = \int_0^T (u_k - u)^2(x, t)dt + 2 \int_0^T v(x, t)[u_\tau(x) - u_\tau(x)](u_k - u)(x, t)dx
\]

\[
+ \sum_{j=1}^2 \int_0^T \left[ \sum_{l=1}^L \kappa_j(x_l, t)(u_k - u)(x_l, t) \right]^2 dt
\]

\[
+ 2 \sum_{j=1}^2 \int_0^T \left[ \sum_{l=1}^L \kappa_j(x_l, t)u(x_l, t) - v_j(t) \right] \left[ \sum_{l=1}^L \kappa_j(x_l, t)(u_k - u)(x_l, t) \right] dt
\]

\[
= I_1^k + I_2^k - I_3^k + I_4^k,
\]

where

\[
I_1^k = \int_0^T \left( \tilde{u}_k - \tilde{\mu}_k * (\tilde{u}_k - \tilde{\mu}) + (\tilde{\mu}_k - \tilde{\mu}) * \tilde{u} \right)^2(x, t)dx,
\]

\[
I_2^k = 2 \int_0^T [u(x, t) - u_\tau(x)](\tilde{u}_k - \tilde{\mu}_k * (\tilde{u}_k - \tilde{\mu}) + (\tilde{\mu}_k - \tilde{\mu}) * \tilde{u})(x, t)dx,
\]

\[
I_3^k = \sum_{j=1}^2 \int_0^T \left[ \sum_{l=1}^L \kappa_j(x_l, t)(\tilde{u}_k - \tilde{u} + \tilde{\mu}_k * (\tilde{u}_k - \tilde{\mu}) + (\tilde{\mu}_k - \tilde{\mu}) * \tilde{u})(x_l, t) \right]^2 dt,
\]
\[ I_k^2 = 2 \sum_{j=1}^{2} \int_0^T \left[ \sum_{l=1}^{L} \kappa_j(x_l, t) u(x_l, t) - v_j(t) \right] \times \left[ \sum_{l=1}^{L} \kappa_j(x_l, t) \left( \bar{u} - \tilde{u} + \bar{\mu} * \left( \bar{u} - \tilde{u} \right) \right) \right] dt. \]

Using the Cauchy inequality, \( \bar{\mu} \in L^2(0, T) \), \( \tilde{u}, \tilde{u} \in \mathcal{U}(\Omega_T) \) and the boundedness of the sequence \( \bar{\mu}_k \) in \( L^2(0, T) \) we obtain
\[ |I_k^2| \leq \left\| \left( \bar{u} - \tilde{u} + \bar{\mu} * (\bar{u} - \tilde{u}) \right)(\cdot, T) \right\|_{L^2(\Omega_T)}^2 + 2\left\| \left( \bar{\mu} - \bar{\mu}_k \right) * \tilde{u} \right\|_{L^2(\Omega_T)} \left\| \left( \bar{u} - \tilde{u} + \bar{\mu}_k * (\bar{u} - \tilde{u}) \right)(\cdot, T) \right\|_{L^2(\Omega_T)} + R_k^2 \]
\[ \leq \bar{C}_1 \left( \left\| \bar{u} - \bar{u}_k \right\|_{L^2(\Omega_T)}^2 + \left\| \bar{u}_k - \bar{u}_k \right\|_{L^2(\Omega_T)} + R_k^2 \right) \]
with a constant \( \bar{C}_1 \) and
\[ R_k^2 = \int_0^\tau \left[ \int_0^\tau \left( \bar{\mu}_k - \bar{\mu} \right)(\tau) \bar{u}(x, T - \tau) d\tau \right] dx. \]

Since \( \tilde{u} \in \mathcal{U}(\Omega_T) \subset L^2(\Omega_T) \), by Tonelli’s theorem it holds \( \tilde{u}(x, \cdot) \in L^2(0, T) \) a.e. \( x \in (c, d) \Rightarrow \tilde{u}(x, T - \cdot) \in L^2(0, T) \) a.e. \( x \in (c, d) \). Thus, in view of \( \mu_k \to \bar{\mu} \) in \( L^2(0, T) \) we have \( \int_0^\tau (\bar{\mu}_k - \bar{\mu})(\tau) \bar{u}(x, T - \tau) d\tau \to 0 \) a.e. \( x \in (c, d) \). Moreover,
\[ \int_0^\tau (\bar{\mu}_k - \bar{\mu})(\tau) \bar{u}(x, T - \tau) d\tau \leq \bar{C}_{11} \int_0^\tau \left| \tilde{u}(x, \tau) \right| d\tau \leq \bar{C}_{11} \left\| \tilde{u}(x, \tau) \right\|_{L^2(\Omega_T)} \]
with a constant \( \bar{C}_{11} \), because the sequence \( \bar{\mu}_k \) is bounded in \( L^2(0, T) \). Therefore, by the dominated convergence theorem we obtain \( R_k^2 \to 0 \). Similarly for \( I_k^2 \) we get
\[ |I_k^2| \leq 2 \left\| u(\cdot, T) - u_T \right\|_{L^2(\Omega_T)} \left\| \left( \bar{u} - \tilde{u} + \bar{\mu}_k * (\bar{u} - \tilde{u}) \right)(\cdot, T) \right\|_{L^2(\Omega_T)} + R_k^2 \]
\[ \leq \bar{C}_2 \left( \left\| \bar{u} - \bar{u}_k \right\|_{L^2(\Omega_T)} + R_k^2 \right), \]
where \( \bar{C}_2 \) is a constant and
\[ st R_k^2 = \int_c^d [u(x, T) - u_T(x)] \int_0^\tau (\bar{\mu}_k - \bar{\mu})(\tau) \bar{u}(x, T - \tau) d\tau dx. \]  \hfill (5.3)

By the same reasons as above, it holds \( R_k^2 \to 0 \). Next, let us estimate \( I_k^2 \):
\[ |I_k^2| \leq L^2 \sum_{j=1}^{2} \max_{1 \leq l \leq L} \left[ \left\| \kappa_j(x_l, \cdot) \right\|_{L^2(0, T)} \left\| \left( \bar{u} - \tilde{u} + \bar{\mu}_k * (\bar{u} - \tilde{u}) \right)(x_l, \cdot) \right\|_{L^2(0, T)} \right] \]
\[ + 2L^2 \sum_{j=1}^{2} \max_{1 \leq l \leq L} \left[ \left\| \kappa_j(x_l, \cdot) \right\|_{L^2(0, T)} \left\| \left( \bar{\mu} - \bar{\mu}_k \right) * \bar{u} \right\|_{L^2(0, T)} \right] \]
\[ \times \left\| \left( \bar{u} - \tilde{u} + \bar{\mu}_k * (\bar{u} - \tilde{u}) \right)(x_l, \cdot) \right\|_{L^2(0, T)} + R_k^2 \]
\[ \leq \bar{C}_3 \left( \left\| \bar{u} - \bar{u}_k \right\|_{L^2(\Omega_T)} + \left\| \bar{u}_k - \bar{u}_k \right\|_{L^2(\Omega_T)} + R_k^2 \right), \]
where \( \bar{C}_3 \) is a constant and
\[ R_k^2 = L^2 \sum_{j=1}^{2} \max_{1 \leq l \leq L} \left\{ \left\| \kappa_j(x_l, \cdot) \right\|_{L^2(0, T)} \int_0^\tau \left[ \int_0^\tau (\bar{\mu}_k - \bar{\mu})(\tau) \bar{u}(x_l, t - \tau) d\tau \right] dt \right\}. \]

Here we also used the embedding \( L^2(\Omega_T) \hookrightarrow L^2((0, T); C[\varepsilon, d]) \) that holds in the case \( n = 1 \). Since \( \bar{u}(x_l, t - \cdot) \in L^2((0, t) \) for all \( t \in (0, T) \) we get \( \int_0^\tau (\bar{\mu}_k - \bar{\mu})(\tau) \bar{u}(x_l, t - \tau) d\tau \)
\( \tau \) drifts to 0 for all \( t \in (0,T) \). Moreover, the sequence \( \left| \int_0^T (\bar{\mu}_k - \bar{\mu})(\tau) \tilde{u}(x_1, t - \tau) d\tau \right| \) is bounded by a constant. Consequently, \( R_k^1 \to 0 \). Analogously we deduce the estimate

\[
|J_k^1| \leq \tilde{C}_4 \| \bar{u}_k - \bar{u} \|_{L^1(0,T)} + R_k^1, \quad \text{where } \tilde{C}_4 \text{ is a constant,}
\]

\[
R_k^1 = 2L \sum_{j=1}^L \left| \sum_{i=1}^L \kappa_{ij}(x_1, t) u(x_i, \cdot) - v_j \right|_{L^2(0,T)} \max_{1 \leq j \leq L} \left\{ \| \kappa_{ij}(x_1, \cdot) \|_{L^\infty(0,T)} \right\}
\times \left[ \int_0^T \left( \int_0^T (\bar{\mu}_k - \bar{\mu})(\tau) \tilde{u}(x_1, t - \tau) d\tau \right)^2 dt \right]^{1/2},
\]

where \( R_k^1 \to 0 \).

Note that if we manage to show that \( \| \bar{u}_k - \bar{u} \|_{L^1(0,T)} \to 0 \) then the proof is complete. Indeed, in this case by virtue of \( R_k^1 \to 0, i = 1, 2, 3, 4 \), from the estimates of \( I_k^1 \) we get \( I_k^1 \to 0, i = 1, 2, 3, 4 \), and due to (5.2) we obtain \( J(\bar{z}_k) \to J(z) \), which implies the statement of the theorem.

As in the proof of Theorem 3.1 we can show that \( \bar{u} \) and \( \bar{u}_k \) are the weak solutions of the following problems:

\[
\bar{u}_t = A\bar{u} - \bar{m} \ast A\bar{u} + f + \phi_x \quad \text{in } \Omega_T, \tag{5.4}
\]

\[
\bar{u} = u_0 \quad \text{in } \Omega \times \{0\}, \tag{5.5}
\]

\[
\bar{u} = \tilde{g} \quad \text{in } \Gamma_1^T, \tag{5.6}
\]

\[
-\nu_A \cdot \nabla \bar{u} + \bar{m} \ast \nu_A \cdot \nabla \bar{u} = \tilde{\nu} \bar{u} + \tilde{\nu} \bar{u} + h + \nu \cdot \phi \quad \text{in } \Gamma_2^T, \tag{5.7}
\]

\[
\bar{u}_{k,t} = A_k \bar{u}_k - \bar{m}_k \ast A_k \bar{u}_k + f + \phi_x \quad \text{in } \Omega_T, \tag{5.8}
\]

\[
\bar{u}_k = u_0 \quad \text{in } \Omega \times \{0\}, \tag{5.9}
\]

\[
\bar{u}_k = \tilde{g}_k \quad \text{in } \Gamma_1^T, \tag{5.10}
\]

\[
-\nu_A \cdot \nabla \bar{u}_k + \bar{m}_k \ast \nu_A \cdot \nabla \bar{u}_k = \tilde{\nu} \bar{u}_k + \tilde{\nu} \bar{u}_k + h + \nu \cdot \phi \quad \text{in } \Gamma_2^T, \tag{5.11}
\]

where \( A_k v = (a_{11} v_x) + a_{kk} v \),

\[
\bar{m} = m - \bar{\mu} + m \ast \bar{\mu}, \quad \bar{m}_k = m_k - \bar{\mu}_k + m_k \ast \bar{\mu}_k,
\]

\[
\bar{g} = g + \mu \ast g, \quad \bar{g}_k = g + \mu_k \ast g.
\]

We now show that \( \bar{m}_k \to \bar{m} \). With any \( \zeta \in L^2(0,T) \) we compute

\[
\langle \bar{m}_k - \bar{m}, \zeta \rangle = \langle m_k - m, \zeta \rangle - \langle \bar{\mu}_k - \bar{\mu}, \zeta \rangle + N_k^1,
\]

\[
N_k^1 = \int_0^T \mu_k(\tau) \int_0^{T-\tau} (m_k - m)(s) \zeta(\tau + s) ds d\tau
\]

\[
+ \int_0^T m(\tau) \int_0^{T-\tau} (\bar{\mu}_k - \bar{\mu})(s) \zeta(\tau + s) ds d\tau.
\]

We use the relations \( m_k \to m, \bar{\mu}_k \to \bar{\mu} \) and treat the term \( N_k^1 \) similarly to the term \( N_k \) in (5.1) to get \( N_k^1 \to 0 \). As a result we get \( \langle \bar{m}_k - \bar{m}, \zeta \rangle \to 0 \), hence \( \bar{m}_k \to \bar{m} \).

Subtracting the problem of \( \bar{u} \) from the problem of \( \bar{u}_k \), we see that \( w_k := \bar{u}_k - \bar{u} \) is a weak solution of the problem

\[
w_{k,t} = A w_k - \bar{m} \ast A w_k + \bar{f}_k + \bar{\phi}_k \quad \text{in } \Omega_T, \tag{5.12}
\]

\[
w = 0 \quad \text{in } \Omega \times \{0\}, \tag{5.13}
\]

\[
w = \bar{g}_k \quad \text{in } \Gamma_1^T, \tag{5.14}
\]
\[-\nu_A \cdot \nabla w_k + \tilde{\mathbf{m}} \ast \nu_A \cdot \nabla w_k = \vartheta w_k + \tilde{h}_k + \nu \cdot \tilde{\phi}_k \quad \text{in} \quad \Gamma_{2, T}, \quad (5.15)\]

where

\[
\begin{align*}
\tilde{f}_k &= (a_k - a)(\tilde{u}_k - \tilde{m}_k \ast \tilde{u}_k) - a(\tilde{m}_k - \tilde{m}) \ast \tilde{u}_k, \\
\tilde{\phi}_k &= -a_{11}(\tilde{m}_k - \tilde{m}) \ast \tilde{u}_k, \\
\tilde{g}_k &= (\mu_k - \mu) \ast g, \\
\tilde{h}_k &= \sigma(\tilde{\mu}_k \ast w_k + (\tilde{\mu}_k - \tilde{\mu}) \ast \tilde{u}).
\end{align*}
\]

To use the weak convergence \(a_k \rightharpoonup a\) in forthcoming estimations we have to introduce the functions \(\rho_k \in W_2^2(c, d)\) being the solutions of the following Neumann problems:

\[
\rho''_k - \rho_k = a_k - a \quad \text{in} \quad (c, d), \quad \rho'_k(c) = \rho'_k(d) = 0.
\]

Then \(\rho_k(x) = \int_c^x G(x, y)(a_k - a)(y)dy, \quad x \in (c, d)\), where

\[
G(x, y) = \frac{1}{2(e^{c-x} - e^{-c})} \begin{cases} 
(e^{c-y} + e^{y-c})(e^{d-x} + e^{x-d}) \quad \text{for} \; y < x \\
(e^{c-y} + e^{y-c})(e^{d-y} + e^{y-d}) \quad \text{for} \; y > x
\end{cases}
\]

is a Green function that satisfies the properties \(G, G_s \in L^\infty(\Omega_T)\). The weak convergence \(a_k \rightharpoonup a\) in \(L^2(c, d)\) implies

\[
\|\rho_k\|_{W_2^2(c, d)} \to 0. \quad (5.16)
\]

Using \(\rho_k\) we rewrite the term \((a_k - a)(\tilde{u}_k - \tilde{m}_k \ast \tilde{u}_k)\) in \(\tilde{f}_k\) as follows:

\[
(a_k - a)(\tilde{u}_k - \tilde{m}_k \ast \tilde{u}_k) = |\rho_k'(\tilde{u}_k - \tilde{m}_k \ast \tilde{u}_k)|_x - \rho_k'(\tilde{u}_k - \tilde{m}_k \ast \tilde{u}_k)_x - \rho_k(\tilde{u}_k - \tilde{m}_k \ast \tilde{u}_k).
\]

According to this relation we change the form of the problem for \(w_k\) as follows:

\[
\begin{align*}
&w_{k, t} = A w_k - \tilde{m} \ast A w_k + \tilde{f}_k + \tilde{\phi}_k \quad \text{in} \quad \Omega_T, \\
&\tilde{u} = 0 \quad \text{in} \quad \Omega \times \{0\}, \\
&\tilde{u} = \tilde{g}_k \quad \text{in} \quad \Gamma_{1, T}, \\
&-\nu_A \cdot \nabla w_k + \tilde{m} \ast \nu_A \cdot \nabla w_k = \vartheta w_k + \tilde{h}_k + \nu \cdot \tilde{\phi}_k \quad \text{in} \quad \Gamma_{2, T}, \quad (5.20)
\end{align*}
\]

where

\[
\tilde{f}_k = -\rho_k'(\tilde{u}_k + \tilde{m}_k \ast \tilde{u}_k)_x - \rho_k(\tilde{u}_k + \tilde{m}_k \ast \tilde{u}_k) - a(\tilde{m}_k - \tilde{m}) \ast \tilde{u}_k, \\
\tilde{\phi}_k = \rho'_k(\tilde{u}_k + \tilde{m}_k \ast \tilde{u}_k) - a_{11}(\tilde{m}_k - \tilde{m}) \ast \tilde{u}_k.
\]

Let \(t\) be an arbitrary number in \([0, T]\). To estimate \(w_k\) we will use the projection operators \(P_t\), defined in (3.22), and \(\overline{P}_t w = \begin{cases} w & \text{in} \quad \Omega_t \\
0 & \text{in} \quad \Omega_T \setminus \Omega_t \end{cases} \quad \text{for} \quad w : \Omega_T \to \mathbb{R}.
\]

Let \(w^k_t\) stand for the weak solution of problem (5.17)–(5.20) with \(\tilde{f}_k, \tilde{g}_k, \tilde{h}_k\) replaced by \(\overline{P}_t \tilde{f}_k, \overline{P}_t \tilde{g}_k, \overline{P}_t \tilde{h}_k\), respectively. Then, due to the causality \(w^k_t = w_k\) in \(\Omega_t\). Applying (3.11) for \(w^k_t\) we obtain

\[
\|w_k\|_{L^2(\Omega_t)} \leq \|w^k_t\|_{L^2(\Omega_t)} \leq \overline{C}_0 \left[ \|\overline{P}_t \tilde{f}_k\|_{L^2(0, T; L^2(c, d))} + \|\overline{P}_t \tilde{g}_k\|_{L^2(0, T; L^2(\Omega_T))} + \|\overline{P}_t \tilde{h}_k\|_{L^2(0, T; L^2(\Gamma_{2, T}))} \right]
\]

\[
= \overline{C}_0 \left[ \|\tilde{f}_k\|_{L^2(0, T; L^2(c, d))} + \|\tilde{g}_k\|_{L^2(0, T; L^2(\Omega_T))} + \|\tilde{h}_k\|_{L^2(0, T; L^2(\Gamma_{2, T}))} \right]
\]

\[
(5.21)
\]
with a constant $\overline{C}_0$. Using the relation $a \in L^2(c, d)$, Cauchy inequality, the inequality (3.21), $g(x, 0) = 0$, the embedding $W^1_2(c, d) \hookrightarrow C[c, d]$ and $\tilde{u}_k = u_k + \tilde{u}$ we estimate:

$$
\|\tilde{f}_k\|_{L^2((0, t), L^1(c, d))} \leq \overline{C}_1 \left[ \|(	ilde{m}_k - \tilde{m}) \ast \tilde{u}_k\|_{L^2((0, t), L^1(c, d))} + \|\rho_k\|_{W^1_2(c, d)}(1 + \|\tilde{m}_k\|_{L^2(0, T)})(\|\tilde{u}_k\|_{L^1(c, d)}) \right]
$$

$$
\leq \overline{C}_1 \left[ \int_0^t \|(	ilde{m}_k - \tilde{m})(t - \tau)\|_{L^2(c, d)} \|u_k\|_{L^2(\Omega_T)} d\tau + \|\rho_k\|_{W^1_2(c, d)}(1 + \|\tilde{m}_k\|_{L^2(0, T)})(\|u_k\|_{L^1(c, d)}) \right] + \overline{R}_k^1,
$$

(5.22)

$$
\|\tilde{g}\|_{L^2(\Omega_T)} \leq \overline{C}_2 \left[ \|(	ilde{m}_k - \tilde{m}) \ast \tilde{u}_{k, x}\|_{L^2(\Omega_T)} + \|\rho_k\|_{W^1_2(c, d)}(1 + \|\tilde{m}_k\|_{L^2(0, T)})(\|\tilde{u}_k\|_{L^2(0, T), C[c, d]}) \right]
$$

$$
\leq \overline{C}_2 \left[ \int_0^t \|(	ilde{m}_k - \tilde{m})(t - \tau)\|_{L^2(c, d)} \|u_{k, x}\|_{L^2(\Omega_T)} d\tau + \|\rho_k\|_{W^1_2(c, d)}(1 + \|\tilde{m}_k\|_{L^2(0, T)})(\|u_k\|_{L^2(0, T), C[c, d]}) \right] + \overline{R}_k^2,
$$

(5.23)

$$
\|\tilde{h}_k\|_{L^2(\Omega_T)} \leq \overline{C}_4 \left[ \|\tilde{\mu}_k \ast u_k\|_{L^2(0, T), W^1_2(c, d)} + \|\tilde{\mu}_k - \tilde{\mu}\| \ast \tilde{u}\|_{L^2(0, T), W^1_2(c, d)} \right]
$$

$$
\leq \overline{C}_4 \int_0^t \|\tilde{\mu}_k(t - \tau)\|_{L^2(0, T), W^1_2(c, d)} d\tau + \overline{R}_k^4,
$$

(5.24)

where $\overline{C}_1, \overline{C}_2, \overline{C}_4$ are constants and

$$
\overline{R}_k^1 = \overline{C}_1 \left[ \|(	ilde{m}_k - \tilde{m}) \ast \tilde{u}\|_{L^2(\Omega_T)} + \|\rho_k\|_{W^1_2(c, d)}(1 + \|\tilde{m}_k\|_{L^2(0, T)})(\|\tilde{u}\|_{L^1(c, d)}) \right],
$$

$$
\overline{R}_k^2 = \overline{C}_2 \left[ \|(	ilde{m}_k - \tilde{m}) \ast \tilde{u}_{x}\|_{L^2(\Omega_T)} + \|\rho_k\|_{W^1_2(c, d)}(1 + \|\tilde{m}_k\|_{L^2(0, T)})(\|\tilde{u}\|_{L^1(c, d)}) \right],
$$

$$
\overline{R}_k^3 = \|\tilde{\mu}_k \ast u_0\|_{L^2(\Omega_T)} + \|\tilde{\mu}_k - \tilde{\mu}\| \ast \tilde{u}\|_{L^2(\Omega_T)} + \|\tilde{\mu}_k \ast u_0\|_{L^2(\Omega_T)},
$$

$$
\overline{R}_k^4 = \overline{C}_4 \left[ \|\tilde{\mu}_k - \tilde{\mu}\| \ast \tilde{u}\|_{L^2(\Omega_T), W^1_2(c, d)} \right].
$$

By the weak convergence $\tilde{m}_k \rightharpoonup \tilde{m}$, $\mu_k \rightharpoonup \mu$, $\tilde{\mu}_k \rightharpoonup \tilde{\mu}$ in $L^2(0, T)$ and the relation $\|\rho_k\|_{W^1_2(c, d)} \to 0$ it holds

$$
\overline{R}_k^j \to 0, \quad j = 1, 2, 3, 4.
$$

(5.26)

Indeed, to prove that $\|z_k \ast \tilde{v}\|_{L^2(\Omega_T)} \to 0$, where $z_k$ is one of the functions $\tilde{m}_k - \tilde{m}$, $\mu_k - \mu$ or $\tilde{\mu}_k - \tilde{\mu}$ and $\tilde{v} \in L^2(\Omega_T)$ is one of the functions $\tilde{u}$, $\tilde{u}_x$, $g_0$, $g_x$ or $g_t$ it is possible to use the dominated convergence theorem, again. More precisely,

$$
\|z_k \ast \tilde{v}\|_{L^2(\Omega_T)} = \left\{ \int_{c}^{d} \int_{0}^{T} \left( \int_{0}^{t} z_k(\tau) \tilde{v}(x, t - \tau) d\tau \right)^2 dx \right\}^{1/2},
$$

where the component $\left( \int_{0}^{t} z_k(\tau) \tilde{v}(x, t - \tau) d\tau \right)^2$ is bounded by an integrable in $x \in (c, d)$ function sup $\|z_k\|_{L^2(0, T)}^2 \|\tilde{v}(x, \cdot)\|_{L^2(0, T)}^2$ and tends to zero for all $t \in (0, T)$ and
a.e. $x \in (c, d)$, because $z_k \to 0$ and $\hat{\psi}(x, t - \cdot) \in L^2(0, T)$ for all $t \in (0, T)$ and a.e. $x \in (c, d)$. (The latter relation follows from $\hat{\psi} \in L^2(\Omega_T)$ and Tonelli’s theorem.) Thus, $\|z_k * \hat{\psi}\|_{L^2(\Omega_T)} \to 0$.

As in proof of Theorem 3.1, we use the norms $\|w\|_{\sigma} = \sup_{0 < t < T} e^{-\sigma t} \|w\|_{U(\Omega_T)}$ with the weights $\sigma \geq 0$ in the space $U(\Omega_T)$. Then in view of (5.22)-(5.25) from (5.21) we deduce

$$
\|w_k\|_{\sigma} \leq \overline{C}_5 \left[ \sup_{0 < t < T} \int_0^t e^{-\sigma(t-\tau)} r_k(t-\tau) e^{-\sigma \tau} \|w_k\|_{U(\Omega_T)} d\tau + \|\rho_k\|_{L^2(0, T)} \|w_k\|_{\sigma} + \sum_{j=1}^4 \overline{R}_k \right]
$$

$$
\leq \overline{C}_5 \left\{ \left( e^{-\sigma t} \|w_k\|_{L^2(0, T)} \right)^T + \|\rho_k\|_{W_{j}^2(0, T)} + \|\hat{\mu}_k\|_{L^2(0, T)} \right\} \|w_k\|_{\sigma} + \sum_{j=1}^4 \overline{R}_k \right],
$$

where $\overline{C}_5$ is a constant and $r_k = |\hat{\mu}_k - \hat{\nu}_k|$. Since $\|e^{-\sigma t} \|_{L^2(0, T)} \to 0$ as $\sigma \to \infty$, $\|\rho_k\|_{W_2(0, T)} \to 0$ and the sequences $\|r_k\|_{L^2(0, T)}$, $\|\hat{\mu}_k\|_{L^2(0, T)}$ are bounded, there exist $\sigma_2 > 0$ and $K_2 \in \mathbb{N}$ such that

$$
\|e^{-\sigma t} \|_{L^2(0, T)} \|r_k\|_{L^2(0, T)} + \|\rho_k\|_{W_2(0, T)}(1 + \|\hat{\mu}_k\|_{L^2(0, T)}) \leq \frac{1}{2\overline{C}_5}
$$

for $k \geq K_2$. This, along with the previous inequality, implies

$$
\|w_k\|_{\sigma} \leq 2\overline{C}_5 \sum_{j=1}^4 \overline{R}_k \quad \text{and hence} \quad \|w_k\|_{U(\Omega_T)} \leq 2e^{\sigma_2 T} \overline{C}_5 \sum_{j=1}^4 \overline{R}_k
$$

for $k \geq K_2$. Taking (5.26) into account we obtain the desired convergence: $\|\hat{u}_k - \hat{\mu}\|_{U(\Omega_T)} = \|w_k\|_{U(\Omega_T)} \to 0$. The theorem is proved.

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**References**


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